06-713: Homework 1
Due September 5 in class

Homework Guidelines

- You are encouraged to collaborate with your classmates on homework problems, with two caveats. First, it is a very good idea for you to make a serious attempt at each problem before you consult with others. Second, once you understand a problem, you are required to write up your own solution. You may not copy part or all of someone else's solution into your own homework.
- Hand in your homework in class on the due date. Late homework will be accepted, by 20% of your grade will be deducted for each day the homework is late. Please make sure that your homework is stapled together and has your name clearly marked on it.
- Most homework assignments will include a number of optional problems. The purpose of these problems is to give you extra exposure and practice with the class material. *You are strongly encouraged to do the optional problems*. However, the optional problems do not have any extra credit associated with them, and they will not be graded.

Required Problems

1. Let Q_n be the set of $2n$ functions $\{\phi_{-n},\ldots,\phi_n\}$ for $-\pi \le x \le \pi$ where

$$\phi_0(x) = 1$$
$$\phi_k(x) = \cos kx \quad \text{for } k = 1,\ldots,n$$
$$\phi_{-k}(x) = \sin(-kx) \quad \text{for } k = -1,\ldots,-n.$$

Also, let F_n define the set of all linear combinations of functions from Q_n.

(i) Show that F_3 is a vector space.

(ii) Not all continuous functions for $-\pi \le x \le \pi$ lie within F_3. Prove this statement by giving an example of a function of this type.

(iii) Show that $(f,g) = \int_{-\pi}^{\pi} f(x)g(x)dx$ is an inner product for F_3.

(iv) Is $(f,g) = \int_{-\pi}^{\pi} \exp(-x^2) f(x)g(x)dx$ an inner product for F_3?

(v) Using the inner product defined in part (iii), define an orthonormal basis for F_2 (not F_3).

2. Let V be the set of all (3×2) matrices. Show that V is a vector space. Also find the dimension of the vector space and give two different bases for the space.

3. (i) Solve $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{A} = \begin{bmatrix} 1 & 5 & 2 & 4 \\ 1 & 0 & -1 & 2 \\ 2 & -4 & -4 & 1 \\ 0 & 2 & 4 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   (ii) For the same $\mathbf{A}$ as in part (i), calculate $\mathbf{A}^4$.

   (iii) For the same $\mathbf{A}$ as in part (i) and $\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, show by direct calculation that $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$.

4. Determine the real values of the parameters p and q for which the following equations have nontrivial solutions.

(i) $\begin{cases} px_1 + 2x_2 - x_3 = 0 \\ 2x_1 - x_2 + 3x_3 = 0 \\ qx_1 - 4x_2 + 2x_3 = 0 \end{cases}$

(ii) $\begin{cases} px_1 + x_2 - 3x_3 = 0 \\ x_1 - 2x_2 - x_3 = 0 \\ x_1 - qx_2 + x_3 = 0 \end{cases}$

Optional Problems

1. Greenberg 10.2.1 (matrix operations)
2. Greenberg 10.2.5 (matrix operations)
3. Greenberg 10.2.24 (matrix operations)
4. Greenberg 10.4.15 (determinants)
5. Greenberg 10.4.17 (determinants)
6. Greenberg 10.4.11 (rank)
7. Greenberg 10.4.17 (rank)
8. In many applications of linear algebra, it is useful to be able to construct an orthogonal basis. In this problem we examine one way of doing this starting with a set of non-orthogonal vectors. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be a set of linearly independent vectors that span an n-dimensional vector space.

(i) Show that these vectors can be used to generate an orthogonal basis for the space by using them to define $\mathbf{v}_i = \mathbf{a}_i - \sum_{j=1}^{i-1} \mathbf{v}_j \frac{(\mathbf{v}_j, \mathbf{a}_i)}{(\mathbf{v}_j, \mathbf{v}_j)}$. [Hint: You have to show that $\mathbf{v}_i$ is orthogonal to *all* $\mathbf{v}_j$ with $j < i$.]

(ii) How can this basis be made orthonormal?

(iii)Use this method and the results of required problem 1 to find an orthonormal basis for F_3.

9. In the following systems of equations, p and q are real numbers. For each system of equations, determine whether a solution exists and whether it can include arbitrary values for one or more of the unknowns, x_i. Your answer to this question may depend on what the values of p and q are.

(i) $\begin{cases} px_1 - x_2 + 3x_3 = 1 \\ x_1 + x_2 - px_3 = q \\ 4x_1 + 3x_2 + x_3 = 0 \end{cases}$

(ii) $\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ -2x_1 + qx_2 - 4x_3 = 0 \\ 3x_2 + x_3 = p \end{cases}$

10. In general, m equilibrium reactions occurring among n chemical species can be described as $\sum_{i=1}^{n} \nu_{ij} A_i = 0$ for $j = 1,\ldots,m$, where A_i is the ith chemical species and ν_{ij} are the stoichiometric coefficients. Note that these equations are linear.

Because the stoichiometry of chemical reactions can be written in various forms, we should consider the rank of the stoichiometric matrix, **A**, in order to determine the number of independent reactions. The number of thermodynamic components is defined by subtracting the number of independent reactions from the number of chemical species. Determine the number of independent reactions and thermodynamic components for the kinetic scheme below, which is relevant to epoxidation of ethylene to ethylene oxide:

$C_2H_4 + ½\ O_2 \rightarrow C_2H_4O$

$C_2H_4 + 3\ O_2 \rightarrow 2\ CO_2 + 2\ H_2O$

$C_2H_4O + 5/2\ O_2 \rightarrow 2\ CO_2 + 2\ H_2O$.