Calculus of Vector Fields

Just like there were three kinds of vector multiplication which can be defined, there are three kinds of differentiation with respect to position.

	Notation	Result
Divergence	$\nabla \cdot \mathbf{v}$	scalar
Curl	$\nabla \times \mathbf{v}$	vector
Gradient	$\nabla \mathbf{v}$	tensor

Shortly, we will provide explicit definitions of these quantities in terms of surface integrals. Let me introduce this type of definition using a more familiar quantity:

Gradient of a Scalar (Explicit)

Recall the previous definition for gradient:

$f = f(\mathbf{r})$: $\quad df = d\mathbf{r} \cdot \nabla f$

Such an implicit definition is like defining $f'(x)$ as that function associated with $f(x)$ which yields:

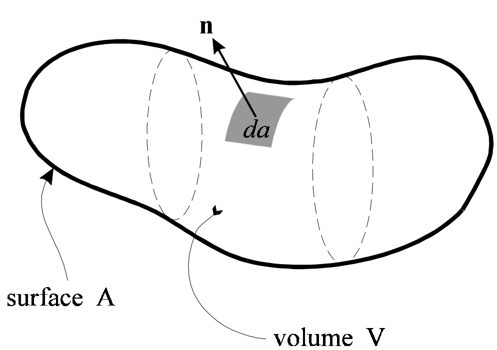
$f = f(x)$: $\quad df = (dx)\, f'$

An equivalent, but explicit, definition of derivative is provided by the Fundamental Theorem of the Calculus:

$$f'(x) \equiv \lim_{\Delta x \to 0} \left\{ \frac{f(x+\Delta x) - f(x)}{\Delta x} \right\} = \frac{df}{dx}$$

We can provide an analogous definition of ∇f

$$\nabla f \equiv \lim_{V \to 0} \left\{ \frac{1}{V} \oint_A \mathbf{n} f da \right\}$$



where f = any scalar field

A = a set of points which constitutes any closed surface enclosing the point $\mathbf{r}$ at which ∇f is to be evaluated

V = volume of region enclosed by A

da = area of a differential element (subset) of A

$\mathbf{n}$ = unit normal to da, pointing out of region enclosed by A

lim ($V \to 0$) = limit as all dimensions of A shrink to zero (in other words, A collapses about the point at which ∇f is to be defined.)

What is meant by this surface integral? Imagine A to be the skin of a potato. To compute the integral:

1) Carve the skin into a number of elements. Each element must be sufficiently small so that

   - element can be considered planar (i.e. $\mathbf{n}$ is practically constant over the element)

   - f is practically constant over the element

2) For each element of skin, compute $\mathbf{n} f\, da$

3) Sum yields integral

This same type of definition can be used for each of the three spatial derivatives of a vector field:

Divergence, Curl, and Gradient

Divergence $$\nabla \cdot \mathbf{v} \equiv \lim_{V \to 0} \left\{ \frac{1}{V} \oint_A \mathbf{n} \cdot \mathbf{v} da \right\}$$

Curl $$\nabla \times \mathbf{v} \equiv \lim_{V \to 0} \left\{ \frac{1}{V} \oint_A \mathbf{n} \times \mathbf{v} da \right\}$$

Gradient $$\nabla \mathbf{v} \equiv \lim_{V \to 0} \left\{ \frac{1}{V} \oint_A \mathbf{n} \mathbf{v} da \right\}$$

Physical Interpretation of Divergence

Let the vector field $\mathbf{v} = \mathbf{v}(\mathbf{r})$ represent the steady-state velocity profile in some 3-D region of space. What is the physical meaning of $\nabla \cdot \mathbf{v}$?

- $\mathbf{n} \cdot \mathbf{v}\, da = dq$ = volumetric flowrate out through da (cm^3/s). This quantity is positive for outflow and negative for inflow.

- $\int_A \mathbf{n} \cdot \mathbf{v}\, da$ = net volumetric flowrate out of enclosed volume (cm^3/s). This is also positive for a net outflow and negative for a net inflow.

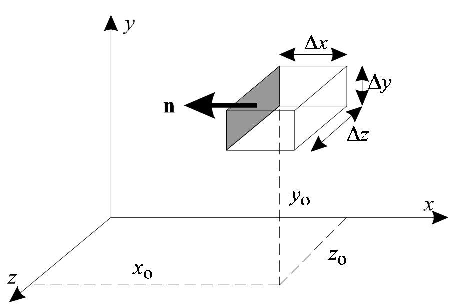
- $(1/V) \int_A \mathbf{n} \cdot \mathbf{v}\, da$ = flowrate out per unit volume (s^{-1})

- $\nabla\bullet\mathbf{v} = \begin{cases} > 0 \text{ for an expanding gas} \\ \quad \text{(perhaps } T\uparrow \text{ or } p\downarrow) \\ = 0 \text{ for an incompressible fluid} \\ \quad \text{(no room for accumulation)} \\ < 0 \text{ for a gas being compressed} \end{cases}$

- $\nabla\cdot\mathbf{v}$ = volumetric rate of expansion of a differential element of fluid per unit volume of that element (s^{-1})

Calculation of $\nabla\cdot v$ *in R.C.C.S.*

Given: $\mathbf{v} = v_x(x,y,z)\mathbf{i} + v_y(x,y,z)\mathbf{j} + v_z(x,y,z)\mathbf{k}$



Evaluate $\nabla\cdot\mathbf{v}$ at (x_o,y_o,z_o).

Solution: Choose A to be surface of rectangular parallelopiped of dimensions $\Delta x,\Delta y,\Delta z$ with one corner at x_o,y_o,z_o.

So we partition A into the six faces of the parallelopiped. The integral will be computed separately over each face:

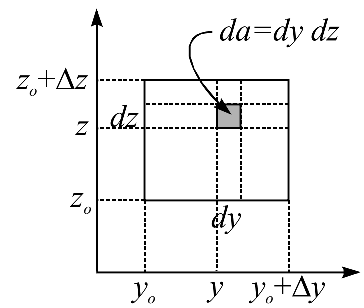
$$\int_A \mathbf{n}\cdot\mathbf{v}\,da = \int_{A_1} \mathbf{n}\cdot\mathbf{v}\,da + \int_{A_2} \mathbf{n}\cdot\mathbf{v}\,da + \cdots + \int_{A_6} \mathbf{n}\cdot\mathbf{v}\,da$$

Surface A_1 is the $x=x_o$ face:

$$\mathbf{n} = -\mathbf{i}$$

$$\mathbf{n}\cdot\mathbf{v} = -\mathbf{i}\cdot\mathbf{v} = -v_x(x_o,y,z)$$

The following figure shows area we will be integrating over



Our differential area patch (shaded) is a rectangle of height dz and width dy. It's area given by

$$da = dy\,dz$$

We multiply this area by the local value of the integrand $-v_x$ and repeat this for each tiny rectangle in the row of width Δy (the integral with respect to y); this is then repeated for each row (the integral with respect to z):

$$\int_{A_1} \mathbf{n}\cdot\mathbf{v}da = \int_{z_o}^{z_o+\Delta z} \int_{y_o}^{y_o+\Delta y} -v_x(x_o,y,z)\,dy\,dz$$

Using the Mean Value Theorem:

$$= -v_x(x_o,y',z')\Delta y\Delta z$$

where $\qquad y_o \le y' \le y_o+\Delta y$

and $\qquad z_o \le z' \le z_o+\Delta z$

Surface A_2 is the $x=x_o+\Delta x$ face:

$$\mathbf{n} = +\mathbf{i}$$

$$\mathbf{n}\cdot\mathbf{v} = \mathbf{i}\cdot\mathbf{v} = v_x(x_o+\Delta x,y,z)$$

$$\int_{A_1} \mathbf{n}\cdot\mathbf{v}\,da = \int_{z_o}^{z_o+\Delta z} \int_{y_o}^{y_o+\Delta y} v_x(x_o+\Delta x,y,z)\,dy\,dz$$

Using the Mean Value Theorem:

$$= v_x(x_o+\Delta x,y'',z'')\Delta y\Delta z$$

where $\qquad y_o \le y'' \le y_o+\Delta y$

and $\qquad z_o \le z'' \le z_o+\Delta z$

The sum of these two integrals is:

$$\int_{A_1} + \int_{A_2} = \left[v_x(x_o+\Delta x,y'',z'') - v_x(x_o,y',z')\right]\Delta y\,\Delta z$$

Dividing by $V = \Delta x \Delta y \Delta z$:

$$\frac{1}{V}\int_{A_1+A_2} \mathbf{n}\cdot\mathbf{v}\,da = \frac{v_x(x_o+\Delta x, y'', z'') - v_x(x_o, y', z')}{\Delta x}$$

Letting Δy and Δz tend to zero:

$$\lim_{\Delta y, \Delta z \to 0}\left\{\frac{1}{V}\int_{A_1+A_2} \mathbf{n}\cdot\mathbf{v}da\right\} =$$

$$= \frac{v_x(x_o+\Delta x, y_o, z_o) - v_x(x_o, y_o, z_o)}{\Delta x}$$

Finally, we take the limit as Δx tends to zero:

$$\lim_{V \to 0}\left\{\frac{1}{V}\int_{A_1+A_2} \mathbf{n}\cdot\mathbf{v}da\right\} = \left.\frac{\partial v_x}{\partial x}\right|_{x_o, y_o, z_o}$$

Similarly, from the two y=const surfaces, we obtain:

$$\lim_{V \to 0}\left\{\frac{1}{V}\int_{A_3+A_4} \mathbf{n}\cdot\mathbf{v}da\right\} = \left.\frac{\partial v_y}{\partial y}\right|_{x_o, y_o, z_o}$$

and from the two z=const surfaces:

$$\lim_{V \to 0}\left\{\frac{1}{V}\int_{A_5+A_6} \mathbf{n}\cdot\mathbf{v}da\right\} = \left.\frac{\partial v_z}{\partial z}\right|_{x_o, y_o, z_o}$$

Summing these three contributions yields the divergence:

$$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Evaluation of **∇×**v *and* **∇**v *in R.C.C.S.*

In the same way, we could use the definition to determine expressions for the curl and the gradient.

$$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}$$

The formula for curl in R.C.C.S. turns out to be expressible as a determinant of a matrix:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}$$

But remember that the determinant is just a mnemonic device, **not the definition** of curl. The gradient of the vector **v** is

$$\nabla\mathbf{v} = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_i}\mathbf{e}_i\mathbf{e}_j$$

where $x_1 = x$, $x_2 = y$, and $x_3 = z$, $v_1 = v_x$, etc.

Evaluation of $\nabla\cdot$v*,* **∇×**v *and* **∇**v *in Curvilinear Coordinates*

Ref: Greenberg, p175

These surface-integral definitions can be applied to any coordinate system. On HWK #2, we obtain $\nabla\cdot\mathbf{v}$ in cylindrical coordinates.

More generally, we can express divergence, curl and gradient in terms of the ***metric coefficients*** for the coordinate systems. If u,v,w are the three scalar coordinate variables for the curvilinear coordinate system, and

$$x = x(u,v,w) \qquad y = y(u,v,w) \qquad z = z(u,v,w)$$

can be determined, then the three metric coefficients — h_1, h_2 and h_3 — are given by

$$h_1(u,v,w) = \sqrt{x_u^2 + y_u^2 + z_u^2}$$

$$h_2(u,v,w) = \sqrt{x_v^2 + y_v^2 + z_v^2}$$

$$h_3(u,v,w) = \sqrt{x_w^2 + y_w^2 + z_w^2}$$

where letter subscripts denote partial differentials while numerical subscripts denote component, and the general expressions for evaluating divergence, curl and gradient are given by

gradient of scalar:

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial v}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial w}\mathbf{e}_3$$

divergence of vector:

$$\nabla\cdot\mathbf{v}=\frac{1}{h_1h_2h_3}\times\left[\frac{\partial}{\partial u}(h_2h_3v_1)+\frac{\partial}{\partial v}(h_1h_3v_2)+\frac{\partial}{\partial w}(h_1h_2v_3)\right]$$

curl of vector:

$$\nabla\times\mathbf{v}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \partial/\partial u & \partial/\partial v & \partial/\partial w\\ h_1v_1 & h_2v_2 & h_3v_3\end{vmatrix}$$

These formulas have been evaluated for a number of common coordinate systems, including R.C.C.S., cylindrical and spherical coordinates. The results are tabulated in Appendix A of BSL (see pages 738-741). These pages are also available online:

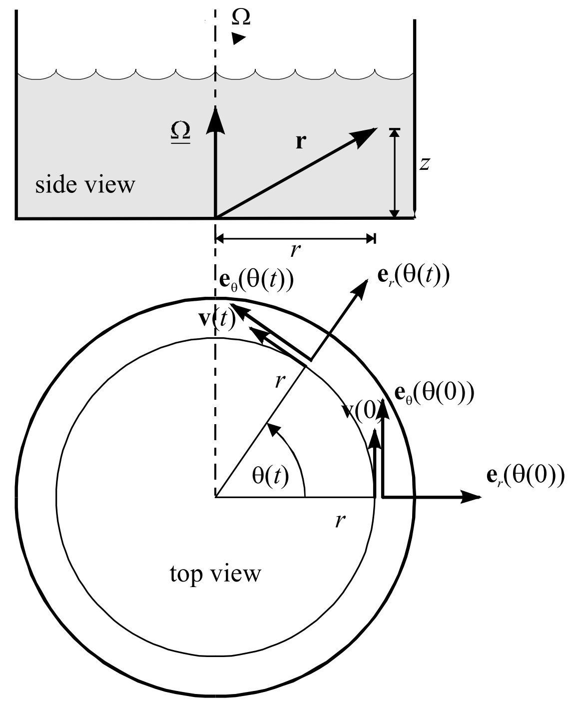
rectangular coords.

cylindrical coords:

spherical coords:

Physical Interpretation of Curl

To obtain a physical interpretation of ∇×**v**, let's consider a particularly simple flow field which is called ***solid-body rotation***. Solid-body rotation is simply the velocity field a solid would experience if it was rotating about some axis. This is also the velocity field eventually found in viscous fluids undergoing steady rotation.



Imagine that we take a container of fluid (like a can of soda pop) and we rotate the can about its axis. After a transient period whose duration depends on the dimensions of the container, the steady-state velocity profile becomes solid-body rotation.

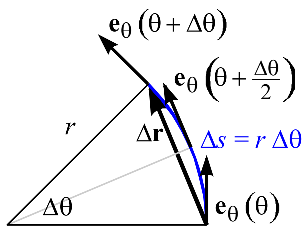
A material point imbedded in a solid would move in a circular orbit at a constant angular speed equal to Ω radians per second. The corresponding velocity is most easily described using cylindrical coordinates with the z-axis oriented perpendicular to the plane of the orbit and passing through the center of the orbit. Then the orbit lies in a z=const plane. The radius of the orbit is the radial coordinate r which is also constant. Only the θ-coordinate changes with time and it increases linearly so that $d\theta/dt$ = const = Ω.

In parametric form in cylindrical coordinates, the trajectory of a material point is given by

$$r(t)=\text{const},\ z(t)=\text{const},\ \theta(t)=\Omega t$$

The velocity can be computed using the formulas developed in the example on page 8:

$$\mathbf{v}=\underbrace{\frac{dr(t)}{dt}}_{0}\mathbf{e}_r+r\underbrace{\frac{d\theta(t)}{dt}}_{r\Omega}\mathbf{e}_\theta+\underbrace{\frac{dz(t)}{dt}}_{0}\mathbf{e}_z=r\Omega\mathbf{e}_\theta$$



Alternatively, we could deduce **v** from the definition of derivative of a vector with respect to a scalar:

$$\mathbf{v} = \frac{D\mathbf{r}}{Dt} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t} = \frac{r\,d\theta\,\mathbf{e}_\theta}{dt} = r\frac{d\theta}{dt}\mathbf{e}_\theta = r\Omega\mathbf{e}_\theta$$

More generally, in invariant form (i.e. in any coordinate system) the velocity profile corresponding to solid-body rotation is given by

$$\mathbf{v}\left(\mathbf{r}_p\right) = \underline{\Omega}\times\mathbf{r}_p \tag{5}$$

where $\underline{\Omega}$ is called the ***angular velocity vector*** and $\mathbf{r}_p$ is the position vector whose origin lies somewhere along the axis of rotation. The magnitude of $\underline{\Omega}$ is the rotation speed in radians per unit time. It's direction is the axis of rotation and the sense is given by the "right-hand rule." In cylindrical coordinates, the angular velocity is

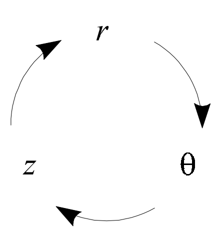
$$\underline{\Omega} = \Omega\mathbf{e}_z$$

and the position vector is $\mathbf{r}_p = r\mathbf{e}_r + z\mathbf{e}_z$

Taking the cross product of these two vectors (keeping the order the same as in (5)):

$$\mathbf{v}(\mathbf{r}) = r\Omega\underbrace{\mathbf{e}_z\times\mathbf{e}_r}_{\mathbf{e}_\theta} + z\Omega\underbrace{\mathbf{e}_z\times\mathbf{e}_z}_{\mathbf{0}} = r\Omega\mathbf{e}_\theta$$

To obtain this result we have used the fact that the cross product of any two parallel vectors vanishes (because the sine of the angle between them is zero — recall definition of cross product on p1).



The cross product of two distinct unit vectors in any right-handed coordinate system yields a vector parallel to the third unit vector with a sense that can be remembered using the figure at right. If the cross product of the two unit vectors corresponds to a "clockwise" direction around this circle, the sense is positive; in a "counter-clockwise" direction, the sense is negative. In this case, we are crossing $\mathbf{e}_z$ with $\mathbf{e}_r$ which is clockwise; hence the cross product is $+\mathbf{e}_\theta$.

Now that we have the velocity field, let's compute the curl. In cylindrical coordinates, the formula for the curl is obtained from p739 of BSL:

$$\nabla\times\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\left(rv_\theta\right)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\mathbf{e}_z$$

Substituting $\quad v_r = 0 \quad v_\theta = r\Omega \quad v_z = 0$

we obtain $\quad \nabla\times\mathbf{v} = 2\Omega\mathbf{e}_z = 2\underline{\mathbf{\Omega}}$

Thus the curl turns out to be twice the angular velocity of the fluid elements. While we have only shown this for a particular flow field, the results turns out to be quite general:

$$\nabla\times\mathbf{v} = 2\underline{\Omega}$$