Vector Field Theory

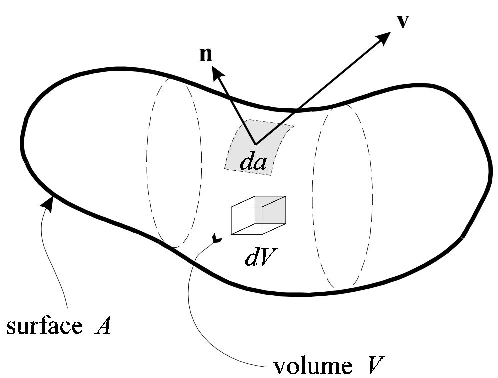
There are three very powerful theorems which constitute "vector field theory:"

- Divergence Theorem
- Stokes Theorem
- Irrotational $\Leftrightarrow$ Conservative $\Leftrightarrow$ Derivable from potential

Divergence Theorem

This is also known as "*Gauss*♣ *Divergence Theorem*" or "*Green's Formula*" (by Landau & Lifshitz). Let **v** be any (continuously differentiable) vector field and choose A to be any (piecewise smooth, orientable) closed surface; then

$$\oint_A \mathbf{n}\cdot\mathbf{v}\, da = \int_V \nabla\cdot\mathbf{v}\, dV$$



where V is the region enclosed by A and **n** is the outward pointing unit normal to the differential surface element having area da.

Although we will not attempt to prove this theorem, we can offer the following rationalization. Consider the limit in which all dimensions of the region are very small, i.e. $V\to 0$. When the region is sufficiently small, the integrand (which is assumed to vary continuously with position)* is just a constant over the region:

$$\nabla\cdot\mathbf{v} = \text{const. inside } V$$

$$\oint_A \mathbf{n}\cdot\mathbf{v}\, da = \int_V \nabla\cdot\mathbf{v}\, dV = (\nabla\cdot\mathbf{v})\left(\int_V dV\right) = (\nabla\cdot\mathbf{v})V$$

Solving for the divergence, we get the definition back (recalling that this was derived for $V\to 0$):

$$\nabla\cdot\mathbf{v} = \frac{1}{V}\oint_A \mathbf{n}\cdot\mathbf{v}\, da$$

Thus the divergence theorem is at least consistent with the definition of divergence.

Corollaries of the Divergence Theorem

Although we have written the Divergence Theorem for vectors (tensors of rank 1), it can also be applied to tensors of other rank:

$$\oint_A \mathbf{n} f\, da = \int_V \nabla f\, dV$$

$$\oint_A \mathbf{n}\cdot\underline{\underline{\tau}}\, da = \int_V \nabla\cdot\underline{\underline{\tau}}\, dV$$

One application of the divergence theorem is to simplify the evaluation of surface or volume integrals. However, we will use GDT mainly to derive invariant forms of the equations of motion:

Invariant: independent of coordinate system.

To illustrate this important application, let's use GDT to derive the continuity equation in invariant form.

The Continuity Equation

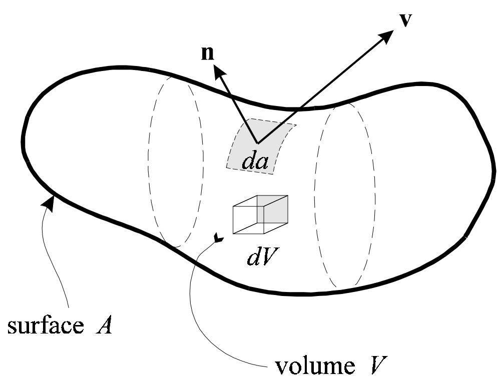
Let $\rho(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$ be the density and fluid velocity. What relationship between them is imposed by conservation of mass?

For any system, conservation of mass means:

♣ Carl Friedrich Gauss (1777-1855), German mathematician, physicist, and astronomer. Considered the greatest mathematician of his time and the equal of Archimedes and Isaac Newton, Gauss made many discoveries before age twenty. Geodetic survey work done for the governments of Hanover and Denmark from 1821 led him to an interest in space curves and surfaces.

* This is a consequence of **v** being "continuously differentiable", which means that all the partial derivatives of all the scalar components of **v** exist and are continuous.

$$\begin{Bmatrix}\text{rate of acc.}\\ \text{of total mass}\end{Bmatrix} = \begin{Bmatrix}\text{net rate of}\\ \text{mass entering}\end{Bmatrix}$$



Let's now apply this principle to an arbitrary system whose boundaries are fixed spatial points (see figure above). Note that this system, denoted by V can be macroscopic (it doesn't have to be differential). The boundaries of the system are the set of fixed spatial points denoted as A. Of course, fluid may readily cross these mathematical boundaries.

Subdividing V into many small volume elements:

$$dm = \rho dV$$

$$M = \int dm = \int_V \rho\, dV$$

$$\frac{dM}{dt} = \frac{d}{dt}\left(\int_V \rho\, dV\right) = \int_V \frac{\partial\rho}{\partial t} dV$$

where we have switched the order of differentiation and integration. This last equality is only valid if the boundaries are independent of t.♥ Now mass enters through the surface A. Subdividing A into small area elements:

$\mathbf{n}$ = outward unit normal

$\mathbf{n}\cdot\mathbf{v}\, da$ = vol. flowrate out through da (cm^3/s)

$\rho(\mathbf{n}\cdot\mathbf{v})da$ = mass flowrate out through da (g/s)

$$\begin{Bmatrix}\text{rate of}\\ \text{mass leaving}\end{Bmatrix} = \oint_A \rho(\mathbf{n}\cdot\mathbf{v})\, da = \int_A \mathbf{n}\cdot(\rho\mathbf{v})\, da$$
$$= \int_V \nabla\cdot(\rho\mathbf{v})\, dV$$

The third equality was obtained by applying GDT. Substituting into the general mass balance:

$$\int_V \frac{\partial\rho}{\partial t} dV = -\int_V \nabla\cdot(\rho\mathbf{v})\, dV$$

Since the two volume integrals have the same limits of integration (same domain), we can combine them:

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] dV = 0$$

Since V is arbitrary, and since this integral must vanish for all V, the integrand must vanish at every point:*

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

which is called the ***equation of continuity***.

Comment: Note that we were able to derive this result in its most general vectorial form, without recourse to any coordinate system and using a **finite** (not differential) control volume. In the special case in which ρ is a constant (i.e. depends on neither time nor position), the continuity equation reduces to:

$$\nabla\cdot\mathbf{v} = 0 \qquad\qquad \rho\text{=const.}$$

Recall that $\nabla\cdot\mathbf{v}$ represents the rate of expansion of fluid elements. "$\nabla\cdot\mathbf{v} = 0$" means that any flow into a fluid element is matched by an equal flow out of the fluid element: accumulation of fluid inside any volume is negligible small.

♥ To see what happens when the boundaries are moving, see the following section entitled "Reynolds Transport Theorem".

* If the domain V were not arbitrary, we would not be able to say the integrand vanishes for every point in the domain. For example:

$$\int_0^{2\pi} \cos\theta\, d\theta = \int_0^{2\pi} \sin\theta\, d\theta = 0$$
$$\int_0^{2\pi} (\cos\theta - \sin\theta)\, d\theta = 0$$

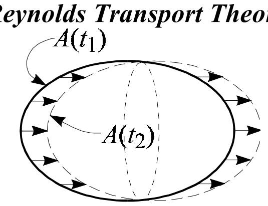
does not imply that cosθ = sinθ since the integral vanishes over certain domains, but not all domains.

Reynolds Transport Theorem

In the derivation above, the boundaries of the system were fixed spatial points. Sometimes it is convenient to choose a system whose boundaries move. Then the accumulation term in the balance will involve time derivatives of volume integrals whose limits change with time. Similar to Liebnitz rule for differentiating an integral whose limits depend on the differentiation variable, it turns out that:♣

$$\frac{d}{dt}\left(\int_{V(t)} S(\mathbf{r},t)dV\right) = \int_{V(t)} \frac{\partial S}{\partial t}dV + \oint_{A(t)} S(\mathbf{r},t)(\mathbf{n}\cdot\mathbf{w})da \tag{6}$$

where **w** is the local velocity of the boundary and $S(t)$ is a tensor of any rank. If $\mathbf{w} = \mathbf{0}$ at all points on the boundary, the boundary is stationary and this equation reduces to that employed in our derivation of the continuity equation. In the special case in which **w** equals the local fluid velocity **v**, this relation is called the ***Reynolds Transport Theorem***.♦



EXAMPLE: rederive the continuity equation using a control volume whose boundaries move with the velocity of the fluid.

Solution: If the boundaries of the system move with the same velocity as local fluid elements, then fluid elements near the boundary can never cross it since the boundary moves with them. Since fluid is not crossing the boundary, the system is ***closed***.* For a closed system, conservation of mass requires:

$$\frac{d}{dt}\begin{Bmatrix}\text{mass of}\\ \text{system}\end{Bmatrix} = 0$$

or

$$\frac{dM}{dt} = \frac{d}{dt}\int_{V(t)} \rho\, dV = 0 \tag{7}$$

Notice that we now have to differentiate a volume integral whose limits of integration depend on the variable with respect to which we are differentiating. Applying (6) with **w**=**v** (i.e. applying the Reynolds Transport Theorem):

$$\frac{d}{dt}\left(\int_{V(t)} \rho(\mathbf{r},t)dV\right) = \int_{V(t)} \frac{\partial \rho}{\partial t}dV + \int_{A(t)} \rho(\mathbf{r},t)(\mathbf{n}\cdot\mathbf{v})da$$

which must vanish by (7). Applying the divergence theorem, we can convert the surface integral into a volume integral. Combining the two volume integrals, we have

$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right]dV = 0$$

which is the same as we had in the previous derivation, except that V is a function of time. However, making this hold for all time and all initial V is really the same as holding for all V. The rest of the derivation is the same as before.

Stokes Theorem

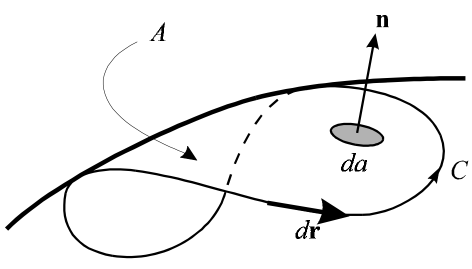
Let **v** be any (continuously differentiable) vector field and choose A to be any (piecewise smooth, orientable) <u>open</u> surface. Then

$$\int_A \mathbf{n}\cdot(\nabla\times\mathbf{v})\,da = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

♣ For a proof, see Greenberg, p163-4.

♦ Osborne Reynolds (1842-1912), Engineer, born in Belfast, Northern Ireland, UK. Best known for his work in hydrodynamics and hydraulics, he greatly improved centrifugal pumps. The Reynolds number takes its name from him.

* When we say "closed," we mean no *net* mass enters or leaves the system; individual molecules might cross the boundary as a result of Brownian motion. However, in the absence of concentration gradients, as many molecules enter the system by Brownian motion as leave it by Brownian motion. **v** is the mass-averaged velocity.



where C is the closed curve forming the edge of A (has direction) and $\mathbf{n}$ is the unit normal to A whose sense is related to the direction of C by the "right-hand rule". The above equation is called ***Stokes Theorem***.♣

Velocity Circulation: Physical Meaning

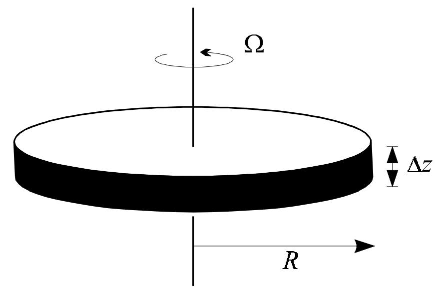
The contour integral appearing in Stokes' Theorem is an important quantity called ***velocity circulation***. We will encounter this quantity in a few lectures when we discuss Kelvin's Theorem. For now, I'd like to use Stokes Theorem to provide some physical meaning to velocity circulation. Using Stokes Theorem and the Mean Value Theorem, we can write the following:

$$\oint_C \mathbf{v}\cdot d\mathbf{r} \overset{\text{Stokes' Theorem}}{=} \int_A \mathbf{n}\cdot(\nabla\times\mathbf{v})\,da$$

$$\overset{\text{Mean Value Theorem}}{=} \langle \mathbf{n}\cdot(\nabla\times\mathbf{v})\rangle A = 2\langle\Omega_n\rangle A$$

Finally, we note from the meaning of curl that $\nabla\times\mathbf{v}$ is twice the angular velocity of fluid elements, so that $\mathbf{n}\cdot\nabla\times\mathbf{v}$ is the normal component of the angular velocity (i.e. normal to the surface A). Thus velocity circulation is twice the average angular speed of fluid elements times the area of the surface whose edge is the closed contour C.

♣ Sir George Gabriel Stokes (1819-1903): British (Irish born) mathematician and physicist, known for his study of hydrodynamics. Lucasian professor of mathematics at Cambridge University 1849-1903 (longest-serving Lucasian professor); president of Royal Society (1885-1890).



Example: Compare "velocity circulation" and "angular momentum" for a thin circular disk of fluid undergoing solid-body rotation about its axis.

Solution: Choosing cylindrical coordinates with the z-axis aligned with axis of rotation. Solid-body rotation corresponds to the following velocity profile (see page 15):

$$\mathbf{v} = r\Omega\mathbf{e}_\theta$$

and $$\nabla\times\mathbf{v} = 2\Omega\mathbf{e}_z$$

Finally the unit normal to the disk surface is $\mathbf{n} = \mathbf{e}_z$. Then the velocity-circulation integral becomes

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = \int_A \mathbf{n}\cdot(\nabla\times\mathbf{v})\,da$$
$$= \int_A \mathbf{e}_z\cdot(2\Omega\mathbf{e}_z)\,da = 2\Omega\pi R^2$$

According to L&L Vol I♣ page 25, the angular momentum $\mathbf{L}$ of a mass m undergoing motion at velocity $\mathbf{v}$ is the lever arm $\mathbf{r}$ times the linear momentum ($\mathbf{p} = m\mathbf{v}$): i.e. $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. Summing this over differential fluid mass in our disk with $dm = \rho\, dV$, the net angular momentum of the disk is:

$$\mathbf{L} = \int_V (\mathbf{r}\times\mathbf{v})\rho\, dV = \rho\Delta z\int_A (\mathbf{r}\times\mathbf{v})\,da$$

Since the disk is of uniform thickness Δz and density ρ, we can write the second equation above in which the volume dV of a cylinder of length Δz and cross-sectional area da is $dV = da\ \Delta z$. If the disk is sufficiently thin that we can neglect the z contribution to the position vector, then we can approximate $\mathbf{r} =$

♣ Landau & Lifshitz, *Mechanics and Electrodynamics* (Course of Theoretical Physics: Vol. 1), Pergamon, 1959.

$r\mathbf{e}_r$ in cylindrical coordinates.♦ Substituting into the second integral above

$$\mathbf{L} = \rho\Delta z \int_A \left(r^2 \Omega \mathbf{e}_z \right) da = \rho\Delta z \Omega \mathbf{e}_z \int_0^R r^2 \, 2\pi r \, dr$$

$$= \frac{\pi}{2} R^4 \rho \Delta z \Omega \mathbf{e}_z$$

Dividing this by the velocity circulation integral:

$$\frac{L_z}{\oint_C \mathbf{v} \cdot d\mathbf{r}} = \frac{\frac{\pi}{2} R^4 \rho \Delta z \Omega}{2\Omega\pi R^2} = \frac{1}{4} R^2 \rho \Delta z = \frac{1}{4\pi} \underbrace{\pi R^2 \Delta z}_{V} \rho$$

$$= \frac{M}{4\pi}$$

where M is the mass of fluid in the disk. This could be rewritten as

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = 4\pi \frac{L_z}{M}$$

So the velocity-circulation integral is just proportional to the angular momentum per unit mass.

Derivable from a Scalar Potential

A very special class of vector fields consists of those vectors for which a scalar field exists such that the vector can be represented as the gradient of the scalar:

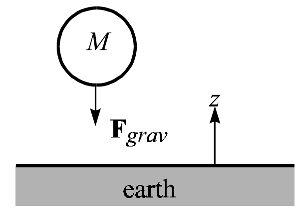
Suppose: $\mathbf{v} = \mathbf{v}(\mathbf{r})$ and $f = f(\mathbf{r})$

If $f(\mathbf{r})$ exists such that:

$$\mathbf{v}(\mathbf{r}) = \nabla f$$

for all $\mathbf{r}$ in some domain, then $f(\mathbf{r})$ is called the ***scalar potential*** of $\mathbf{v}$ and $\mathbf{v}$ is said to be ***derivable from a potential*** in that domain.

♦ Actually this assumption isn't necessary since any z-component of $\mathbf{r}$ will produce an r-component in the cross-product and this r-component will integrate to zero as long as V is a body of rotation about the same axis.



An example of a vector field which is "derivable from a potential" is the gravitational force near sea level:

$$\mathbf{F}_{\text{grav}} = -Mg\mathbf{k}$$

and the associated potential energy is:

$$\phi(z) = Mgz$$

Note that $\nabla\phi = Mg\mathbf{k}$

is identical to the force, except for the sign (introduced by convention). This example also suggests why ϕ is called the "potential" of $\mathbf{v}$. Not every vector field has a potential. Which do? To answer this, let's look at some special properties of such vector fields.

Property I: if $\mathbf{v}=\nabla\phi$ then $\nabla\times\mathbf{v}=\mathbf{0}$

Proof: Recall that $\nabla\times(\nabla\phi) = \mathbf{0}$ (see HWK #2, Prob. 5e). A vector whose curl vanishes throughout some region is said to be ***irrotational***. This name is an allusion to $\nabla\times\mathbf{v}$ representing the rotation rate if $\mathbf{v}$ is the fluid velocity. $\nabla\times\mathbf{v}=\mathbf{0}$ means the fluid elements are not rotating.

Property II: if $\mathbf{v}=\nabla\phi$ then $\oint_C \mathbf{v} \cdot d\mathbf{r} = 0$

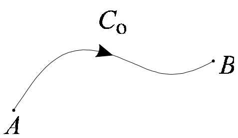
for any closed contour in the region.

Proof: Using Property I, we know that $\nabla\times\mathbf{v}=\mathbf{0}$. Then we can deduce the value of this closed-contour integral from Stokes' Theorem:

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_A \mathbf{n} \cdot \underbrace{(\nabla \times \mathbf{v})}_{\mathbf{0}} da = \mathbf{0}$$

A vector field which has this property is said to be ***conservative***. This name is an allusion to the special case in which $\mathbf{v}$ represents a force, like gravity. Then $\mathbf{v}\cdot d\mathbf{r}$ (force times displacement) represents the work required to move the object through the force field. Saying that the contour integral vanishes means that the work required to lift a weight can be recovered when the weight falls. In other words, energy is conserved.

If C is open, $\mathbf{v}=\nabla\phi$ is still quite useful:



Property III: let C_o be an **open** contour connecting points A and B.

If $\mathbf{v}=\nabla\phi$ then $\int_{C_o}\mathbf{v}\cdot d\mathbf{r} = \phi(\mathbf{r}_B)-\phi(\mathbf{r}_A)$

for any contour connecting A and B.

Proof: Note that $\nabla\phi\cdot d\mathbf{r} = d\phi$ (from our definition of gradient). Then

$$\int_{C_o} \mathbf{v}\cdot d\mathbf{r} = \int_{C_o} d\phi = \phi(\mathbf{r}_B) - \phi(\mathbf{r}_A)$$

We call this property ***path independence*** since any path connecting A and B gives the same result. Of course, Property II is just a special case of this for which A=B so that $\phi(\mathbf{r}_B) - \phi(\mathbf{r}_A) = 0$.

Theorem III

We have just shown that properties I and II are implied by $\mathbf{v} = \nabla\phi$; it turns out that the converse is also true, although I'm not going to prove it here. We can distill these properties and their converse into a single statement:

$$\left\{\begin{array}{c}\nabla\times\mathbf{v}=\mathbf{0}\\ \text{for all } \mathbf{r}\\ \text{in Region}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\phi=\phi(\mathbf{r}) \text{ exists}\\ \text{such that } \mathbf{v}=\nabla\phi\\ \text{in Region}\end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{c}\oint_C \mathbf{v}\cdot d\mathbf{r} = 0 \text{ for every}\\ \text{closed } C \text{ in Region}\end{array}\right\}$$

which we will refer to as "Theorem III".

Transpose of a Tensor, Identity Tensor

The ***transpose*** of a tensor $\underline{\underline{\tau}}$ is denoted $\underline{\underline{\tau}}^t$ and is defined so that:

$$\mathbf{v}\cdot\underline{\underline{\tau}} = \underline{\underline{\tau}}^t\cdot\mathbf{v}$$

and $$\underline{\underline{\tau}}\cdot\mathbf{v} = \mathbf{v}\cdot\underline{\underline{\tau}}^t$$

for all vectors $\mathbf{v}$. For example:

if $$\underline{\underline{\tau}} = \mathbf{ab}$$

then $$\underline{\underline{\tau}}^t = \mathbf{ba}$$

More generally, in terms of scalar components of $\underline{\underline{\tau}}$, we can write the relationship between a tensor and its transpose as:

$$\tau^t_{ij} = \tau_{ji}$$

Symmetric Tensor: $\underline{\underline{\tau}}^t = \underline{\underline{\tau}}$

An example of a symmetric tensor is the dyad **aa**.

Identity Tensor: Also known as the ***Idem Factor***. Denoted as $\underline{\underline{\mathbf{I}}}$ and defined so that:

$$\mathbf{v}\cdot\underline{\underline{\mathbf{I}}} = \mathbf{v} = \underline{\underline{\mathbf{I}}}\cdot\mathbf{v}$$

for any vector $\mathbf{v}$. Clearly $\underline{\underline{\mathbf{I}}}$ is symmetric, but in addition, dotting it with another vector gives that vector back (like multiplying by one). In any coordinate system, $\underline{\underline{\mathbf{I}}}$ can be calculated from:

$$\underline{\underline{\mathbf{I}}} = \nabla\mathbf{r} = \frac{\partial\mathbf{r}}{\partial\mathbf{r}}$$

where $\mathbf{r}$ is the position vector expressed in terms of the unit vectors in that coordinate system. Recalling from p3 that gradient can be thought of as the partial derivative with respect to position, $\underline{\underline{\mathbf{I}}}$ can be thought of as the derivative of the position vector with respect to itself. In R.C.C.S., recall that:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and $$\nabla\mathbf{r} = \sum_i\sum_j \frac{\partial r_j}{\partial x_i}\mathbf{e}_i\mathbf{e}_j$$

where r_j is the j^{th} component of the position vector $\mathbf{r}$ and x_i is the i^{th} coordinate. In Cartesian coordinates, the position vector components are related to the coordinates according to:

$$r_1 = x_1 = x,\ r_2 = x_2 = y, \text{ and } r_3 = x_3 = z:$$

then $$\frac{\partial r_j}{\partial x_i} = \delta_{ij}$$

which is 0 if $i\neq j$ or 1 if $i=j$. This leaves:

$$\nabla \mathbf{r} = \sum_i \sum_j \delta_{ij} \mathbf{e}_i \mathbf{e}_j = \sum_i \mathbf{e}_i \mathbf{e}_i \, ^*$$

so $$\underline{\underline{\mathbf{I}}} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$$

As a partial proof that $\underline{\underline{\mathbf{I}}}$ has the desired properties which make it the identity tensor, consider dotting it with an arbitrary vector **v**:

$$\mathbf{v}\cdot(\mathbf{ii}+\mathbf{jj}+\mathbf{kk}) = \mathbf{v}\cdot\mathbf{ii} + \mathbf{v}\cdot\mathbf{jj} + \mathbf{v}\cdot\mathbf{kk}$$

$$= (\mathbf{v}\cdot\mathbf{i})\mathbf{i} + (\mathbf{v}\cdot\mathbf{j})\mathbf{j} + (\mathbf{v}\cdot\mathbf{k})\mathbf{k}$$

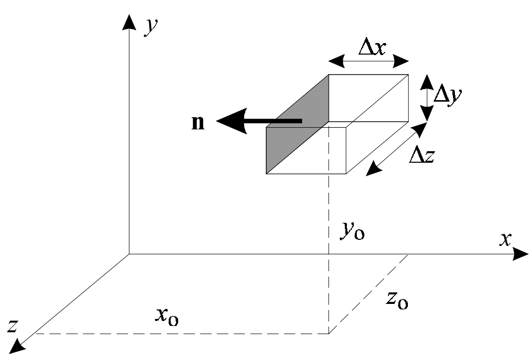
$$= v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = \mathbf{v}$$

Thus we have shown that $\mathbf{v}\cdot\underline{\underline{\mathbf{I}}}=\mathbf{v}$, as advertised.

Divergence of a Tensor

In presenting the corollaries to the Divergence Theorem, we have already introduced the divergence of a tensor. This quantity is defined just like divergence of a vector.

$$\nabla\cdot\underline{\underline{\tau}} \equiv \lim_{V\to 0}\left\{\frac{1}{V}\int_A \mathbf{n}\cdot\underline{\underline{\tau}}\,da\right\}$$



Note that this definition uses a pre-dot not a post-dot. In R.C.C.S.

$$\underline{\underline{\tau}} = \Sigma\Sigma\tau_{ij}\mathbf{e}_i\mathbf{e}_j$$

On the x=x_o face:

$$\mathbf{n} = -\mathbf{e}_1$$

$$\mathbf{n}\cdot\underline{\underline{\tau}} = \sum_i \sum_j \tau_{ij}\mathbf{e}_1\cdot\mathbf{e}_i\mathbf{e}_j = -\sum_j \tau_{1j}\Big|_{x_o}\mathbf{e}_j$$

Similarly, on the x=$x_o+\Delta x$ face, we obtain:

$$\mathbf{n} = +\mathbf{e}_1$$

$$\mathbf{n}\cdot\underline{\underline{\tau}} = \sum_j \tau_{1j}\Big|_{x_o+\Delta x}\mathbf{e}_j$$

After integrating over the area, we obtain:

$$\int_{A_1+A_2} \mathbf{n}\cdot\underline{\underline{\tau}}\,da = \sum_j\left\{\tau_{1j}\,\big|_{x_o+\Delta x} - \tau_{1j}\,\big|_{x_o}\right\}\mathbf{e}_j\,\Delta y\Delta z$$

Dividing by V:

$$\frac{1}{V}\int_{A_1+A_2}\mathbf{n}\cdot\underline{\underline{\tau}}\,da = \sum_j\left\{\frac{\tau_{1j}\,\big|_{x_o+\Delta x} - \tau_{1j}\,\big|_{x_o}}{\Delta x}\right\}\mathbf{e}_j$$

Taking the limit as $V\to 0$:

$$\lim_{V\to 0}\left\{\frac{1}{V}\int_{A_1+A_2}\mathbf{n}\cdot\underline{\underline{\tau}}\,da\right\} = \sum_j \frac{\partial\tau_{1j}}{\partial x}\mathbf{e_j}$$

Adding on similar contributions from the y=const and z=const faces:

$$\nabla\cdot\underline{\underline{\tau}} = \sum_i\sum_j \frac{\partial\tau_{ij}}{\partial x_i}\mathbf{e}_j$$

$$\nabla\cdot\underline{\underline{\tau}} = \left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)\mathbf{i} + \left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)\mathbf{j} + \left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)\mathbf{k}$$

* This expression for the identity tensor is valid for any set of orthonormal unit vectors (not just the Cartesian ones for which we have derived it here).