Stream Function

Before we resolve d’Alembert’s paradox by adding viscous forces, let’s step back for a moment and review what we have accomplished for potential (or irrotational) flow. The mathematical problem might be stated as: find \( \mathbf{v}(x, t) \) such that:

\[
\nabla \times \mathbf{v} = \mathbf{0} \tag{18}
\]

and

\[
\nabla \cdot \mathbf{v} = 0 \tag{19}
\]

Eqs. (18) and (19) represent four partial differential equations in the components of the unknown vector field. Recognizing that the solution is derivable from a potential allows us compress these four equations into one scalar equation in one unknown:

\[
\mathbf{v} = \nabla \phi; \quad \nabla^2 \phi = 0
\]

which is quite a remarkable trick. The potential is not the only scalar field which a vector field can be expressed in terms of. Velocity can also be expressed in terms of a stream function.

Potential (\( \phi \)) — a scalar field whose relationship to \( \mathbf{v} \) is carefully selected to automatically satisfy irrotationality

\[
\mathbf{v} = \nabla \phi \rightarrow \nabla \times \mathbf{v} = \mathbf{0}
\]

Whereas the relation between velocity and scalar potential is chosen to automatically satisfy (18), the relationship between velocity and stream function is chosen to automatically satisfy (19):

\[
\mathbf{v} = \mathbf{f}(\psi) \rightarrow \nabla \cdot \mathbf{v} = 0
\]

Stream Function (\( \psi \)) — a scalar field whose relationship to \( \mathbf{v} \) is carefully selected to automatically satisfy continuity.

It turns out that it is sufficient to express \( \mathbf{v} \) as the curl of another vector. According to Identity C.6, the divergence of the curl of any vector is zero (See HWK #2, Prob. 5d):

\[
\nabla \times [\nabla \times (\psi \mathbf{e}_z)] = 0 \quad \text{identity C.6}
\]

\[
\mathbf{v} = \nabla \times \mathbf{u} \rightarrow \nabla \cdot \mathbf{v} = 0
\]

A vector which can be expressed as the curl of another vector is said to be solenoidal. \( \mathbf{u} \) is called the vector potential of \( \mathbf{v} \). Of course, knowing that \( \mathbf{v} = \nabla \times \mathbf{u} \) isn’t always of much help because we just trade one unknown vector for another. Fortunately, there are several broad classes of flows for which the form of the vector potential is known.

Two-D Flows

When nothing happens along one of the three directions in R.C.C.S., we have 2-D flow:

\[
\mathbf{v} = v_x(x, y)\mathbf{i} + v_y(x, y)\mathbf{j}
\]

or

\[
v_z = 0, \quad \partial v_z / \partial z = 0
\]

For such a flow,\

\[
\mathbf{v} = \nabla \times [\psi(x, y)\mathbf{e}_z] = \nabla \psi \times \mathbf{e}_z + \psi \nabla \times \mathbf{e}_z
\]

\[
= \left( \frac{\partial \psi}{\partial x} \mathbf{e}_x + \frac{\partial \psi}{\partial y} \mathbf{e}_y \right) \times \mathbf{e}_z
\]

\[
= \frac{\partial \psi}{\partial x} \mathbf{e}_x \times \mathbf{e}_z + \frac{\partial \psi}{\partial y} \mathbf{e}_y \times \mathbf{e}_z
\]

\[
= -\frac{\partial \psi}{\partial y} \mathbf{e}_x
\]

In terms of its scalar components, the velocity is:

\[

\begin{align*}
\frac{\partial \psi}{\partial x} & \quad \quad \frac{\partial \psi}{\partial y} \quad \quad \frac{\partial \psi}{\partial z} = 0 \tag{21}
\end{align*}
\]

Next we substitute this form for \( \mathbf{v} \) into (19):

\[
\nabla \cdot \mathbf{v} = \nabla \cdot \left[ \nabla \times (\psi \mathbf{e}_z) \right] = 0 \quad \text{identity E.3}
\]

which automatically satisfies continuity (19), for any choice of \( \psi(x, y) \). The scalar field \( \psi(x, y) \) is called the stream function. For irrotational flow, the problem would be to determine \( \psi(x, y) \) such that (18) is satisfied:

\[
\nabla \times \left[ \nabla \times (\psi \mathbf{e}_z) \right] = 0
\]

* Although it might appear that we have overspecified the problem by specifying both the divergence and curl (which represent four scalar equations in the three components of \( \mathbf{v} \)), this turns out not to be true. In general, both the divergence and curl must be specified throughout some region in space before the vector field can be determined in that region.

* “Identity E.3” in this equation refers to one of the mathematical identities summarized on the handout titled “Useful Identities in Vector Notation”.

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We can reduce this to a scalar equation. Using identity E.5 from handout:

\[ \nabla \times \left( \nabla \times (\psi e_z) \right) = \nabla \left( \nabla \cdot (\psi e_z) \right) - \nabla^2 (\psi e_z) \]

but \[ \nabla \cdot (\psi e_z) = \frac{\partial \psi}{\partial z} = 0 \]

and \[ \nabla^2 (\psi e_z) = (\nabla^2 \psi) e_z \]

Thus \[ \nabla \times \mathbf{v} = - (\nabla^2 \psi) e_z \]

So for irrotational flow, the streamfunction must also satisfy Laplace's equation:

\[ \nabla \times \mathbf{v} = \mathbf{0}; \quad \nabla^2 \psi = 0 \]

Unlike the scalar potential, the streamfunction can be used in all 2-D flows, including those for which the flow is not irrotational. Indeed, we will use the streamfunction to solve Stokes flow of a viscous fluid around a sphere, in which the fluid is not even ideal.

**Axisymmetric Flow (Cylindrical)**

Another general class of flows for which a streamfunction exists is axisymmetric flow. In cylindrical coordinates \((r, \theta, z)\), this corresponds to:

\[ \mathbf{v} = v_r(r, z) \mathbf{e}_r + v_z(r, z) \mathbf{e}_z \]

or \[ v_\theta = 0 \] and \( \partial / \partial \theta = 0 \)

Then \( \nabla \cdot \mathbf{v} = 0 \) can be satisfied by seeking \( \mathbf{v} \) of the form:

\[ \mathbf{v} = \nabla \times \left[ f(r, z) \mathbf{e}_\theta \right] \]

or \[ \mathbf{v} = \nabla \times \left[ \frac{\psi(r, z)}{r} \mathbf{e}_\theta \right] \]

The second expression usually leads to somewhat simpler expressions for \( \nabla \times \mathbf{v} \) and is the one used in the table on p131 of BS&L:

\[ v_r = - \frac{1}{r} \frac{\partial \psi}{\partial z} \quad v_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \]

Computing the curl in cylindrical coordinates and setting it equal to zero leads to the following PDE in \( \psi \) (the details are left as an exercise):

\[ \nabla \times \mathbf{v} = - \frac{E^2 \psi}{r} \mathbf{e}_\theta = \mathbf{0} \quad \longrightarrow \quad E^2 \psi = 0 \]

where \( E^2 \psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} \)

Note that \( E^2 \psi \neq \nabla^2 \psi \):

\[ \nabla^2 \psi(r, z) = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} \]

**Axisymmetric Flow (Spherical)**

In spherical coordinates \((r, \theta, \phi)\), axisymmetric flow means

\[ \mathbf{v} = v_r(r, \theta) \mathbf{e}_r + v_\theta(r, \theta) \mathbf{e}_\theta \]

or \( v_\phi = 0 \) and \( \partial / \partial \phi = 0 \)

where \( \phi \) is the azimuthal angle. Then \( \nabla \cdot \mathbf{v} = 0 \) can be satisfied by seeking \( \mathbf{v} \) of the form:

\[ \mathbf{v} = \nabla \times [ \psi'(r, \theta) \mathbf{e}_\phi ] \]

or

\[ \mathbf{v} = \nabla \times \left[ \frac{\psi(r, \theta)}{r \sin \theta} \mathbf{e}_\phi \right] \]

Again, the second expression is the one used in the table on p131 of BS&L (except the signs are reversed). Taking the curl (HWK #6, Prob. 1a):

\[ \nabla \times \mathbf{v} = - \frac{\mathbf{e}_\phi}{r \sin \theta} E^2 \psi = \mathbf{0} \]

which requires \( E^2 \psi = 0 \)

where \( E^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial \psi}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) \)

Note that \( E^2 \psi \neq \nabla^2 \psi \):

\[ \nabla^2 \psi(r, \theta) = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) \]

**Orthogonality of \( \psi=\text{const} \) and \( \phi=\text{const} \)**

A contour on which \( \psi=\text{const} \) is called a **streamline**. A contour on which \( \phi=\text{const} \) is called a **iso-potential line**. It turns out that these two contours are orthogonal at any point in the fluid. To see this first note that

\[ \mathbf{v} = \nabla \phi \]

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From the geometric meaning of gradient (see page 3 of Notes), we know that $\nabla \phi$ and hence $\mathbf{v}$ is normal to a $\phi=$const surface (see figure at right). Second, recall that $\mathbf{v}$ can also be written in terms of streamfunction as
for R.C.C.S.
$$\mathbf{v} = \nabla \psi \times \mathbf{k}$$  \hspace{1cm} (22)

Recall that the cross product is a vector which is orthogonal to two vectors being multiplied. Thus $\mathbf{v}$, $\nabla \psi$, and $\mathbf{k}$ are mutually orthogonal. Since $\mathbf{v}$ and $\nabla \phi$ point in the same direction, $\nabla \phi$ and $\nabla \psi$ must also be orthogonal.

**Streamlines, Pathlines and Streaklines**

**Streamline** - a contour in the fluid whose tangent is everywhere parallel to $\mathbf{v}$ at a given instant of time.

**Path Line** - trajectory swept out by a fluid element.

**Streak Line** - a contour on which lie all fluid elements which earlier past through a given point in space (e.g. dye trace)

For steady flows, these three definitions describe the same contour but, more generally, they are different.

**Physical Meaning of Streamfunction**

The precise meaning of streamfunction is somewhat different for 2-D and axisymmetric flows. Let's focus on 2-D flows normal to a cylinder (not necessarily with circular cross-section, axis corresponds to $z$-axis in R.C.C.S.). To motivate the somewhat lengthy analysis which follows, we first state the physical meaning. First we observe that material points follow trajectories which can be described as $\psi=$const. Three such trajectories are shown at right which lie in the $xy$-plane. When these trajectories are (mathematically) translated along the $z$-axis a distance $L$ they sweep-out $\psi=$const surfaces.

No fluid crosses these surfaces: there are like the walls of a tube. Since no fluid leaves or enters this “tube”, conservation of mass means the mass flowrate must be a constant at any point along the tube. For an incompressible fluid, the volumetric flowrate is also constant. Suppose we denote the volumetric flowrate between any two of these $\psi=$const surfaces as $\Delta Q$; then it turns out that

$$\frac{\Delta Q}{L} = \psi_2 - \psi_1$$

Thus $\psi$ might be interpreted as the volumetric flowrate per unit length between this particular streaming surface and the one corresponding to $\psi=0$.

Now let's show this. Consider an arbitrary open contour ($C$) in the $xy$-plane, cutting across the flow. Next, consider the surface ($A$) formed by translating this contour a distance $L$ parallel to the $z$-axis. The volumetric flowrate across $A$ is:

$$Q = \int_A \mathbf{n} \cdot \mathbf{v} \, da$$

where $\mathbf{n}$ is a unit normal to $A$. Since nothing changes with $z$, we choose a short segment of the contour, having length $ds$ and of length $L$ as our differential area element.

$$da = L \, ds,$$
The flowrate becomes:

$$\frac{Q}{L} = \int_{C} \mathbf{n} \cdot \mathbf{v} \ ds = \int_{C} \nabla \psi \cdot d\mathbf{r} \tag{23}$$

where \( \mathbf{n} \) is now normal to \( C \) and lies in the \( xy \)-plane. The key to this proof is deriving the second equality in the expression above.

First, \( \mathbf{v} \) can be expressed in terms of the gradient of the streamfunction using \( (20) \) on page 37:

$$\mathbf{v} = \nabla \psi \times \mathbf{k} \tag{20}$$

Both \( \mathbf{v} \) and \( \nabla \psi \) lie in the plane of the page, whereas \( \mathbf{k} \) is perpendicular to this plane (points out of page). Post-crossing \( \nabla \psi \) by the unit vector \( \mathbf{k} \) does not change the magnitude but rotates \( \nabla \psi \) by 90° clockwise. If instead, we pre-crossed by \( \mathbf{k} \) we would rotate \( \nabla \psi \) by 90° counter-clockwise. In either case, the cross product of \( \mathbf{k} \) and \( \nabla \psi \) is a vector lying in the plane of the page and of the same magnitude as \( \nabla \psi \).

The other term in the integrand of \( (23) \) is \( \mathbf{n} \ ds \), where \( ds \) is the magnitude of a differential displacement along the contour, which we will call \( d\mathbf{r} \):

$$ds = |d\mathbf{r}|$$

Since \( \mathbf{n} \) is a unit vector \( \mathbf{n} \ ds \) has the same magnitude as \( d\mathbf{r} \) but is rotated by 90°. Both \( \mathbf{n} \ ds \) and \( d\mathbf{r} \) lie in the plane of the page. Just as in \( (20) \), we can rotate one vector into the other by crossing with \( \mathbf{k} \):

$$d\mathbf{r} \times \mathbf{k} = \mathbf{n} \ ds \tag{24}$$

Substituting \( (20) \) and \( (24) \) into \( (23) \) yields

$$\mathbf{n} \cdot \mathbf{v} \ ds = \mathbf{v} \cdot \mathbf{n} \ ds = \left( \nabla \psi \times \mathbf{k} \right) \cdot (d\mathbf{r} \times \mathbf{k}) = \nabla \psi \cdot d\mathbf{r} = d\psi \tag{25}$$

The 3rd equality above says that the dot product of the two rotated vectors is the same as the dot product of the two vectors without rotation (since they are both rotated by the same amount). \( (25) \) into \( (23) \) yields:

$$\frac{Q}{L} = \int_{C} \mathbf{n} \cdot \mathbf{v} \ ds = \int_{C} d\psi = \psi(Q) - \psi(P)$$

To extract the physical meaning of this results, consider two contours, denoted as \( C_1 \) and \( C_2 \) in the figure at right. Notice that if \( C_1 \) coincides with a streamline, the velocity is parallel to the contour at every point any no fluid crosses it:

then:

$$Q/L = 0$$

and

$$\psi(Q) = \psi(P) = \psi_2$$

Thus

$$\psi = \text{const. along a streamline}$$

On the other hand, if the contour cuts across two streamlines (see contour \( C_2 \) in figure at right), then the difference in value of \( \psi \) corresponding to two different streamlines is just the volumetric flowrate of fluid held between the two streamlines (per unit length in the \( z \)-direction):

$$\Delta \psi = \Delta Q/L$$

**Incompressible Fluids**

By “incompressible fluid” we are usually referring to the assumption that the fluid’s density is not a significant function of time or of position. In other words,
\[
\frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

can be replaced by \( \nabla \cdot \mathbf{v} = 0 \)

For steady flows, \( \frac{\partial p}{\partial t} = 0 \) already and the main further requirement is that density gradients be negligible

Identity C.1
\[
\nabla \cdot (\rho \mathbf{v}) = \rho (\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \rho \approx \rho (\nabla \cdot \mathbf{v})
\]

Since flow causes the pressure to change, we might expect the fluid density to change — at least for gases. As we shall see shortly, gases as well as liquids can be treated as incompressible for some kinds of flow problems. Conversely, in other flow problems, neither gas nor liquid can be treated as incompressible. So what is the real criteria?

For an ideal fluid (i.e. no viscous dissipation to cause \( \nabla T \)), density variations come about primarily because of pressure variations. For an isentropic expansion, the compressibility of the fluid turns out to be:

\[
\left( \frac{\partial p}{\partial p} \right)_S = \frac{1}{c^2}
\]

where \( c \) = speed of sound in the fluid

Thus changes in density caused by changes in pressure can be estimated as

\[
\Delta \rho \approx \frac{1}{c^2} \Delta p
\]

(26)

According to Bernoulli’s equation, pressure changes for steady flow are related to velocity changes:

\[
\frac{p}{\rho} + \frac{v^2}{2} = \text{const.} \quad \text{or} \quad \Delta p = -\frac{1}{2} \rho \Delta v^2
\]

(27)

(27) into (26):

\[
\Delta \rho \approx -\frac{\rho \Delta v^2}{2c^2}
\]

The largest change in density corresponds to the largest change in \( v^2 \), which is \( v_{\text{max}}^2 - 0 \):

\[
\left( \frac{\Delta \rho}{\rho} \right)_{\text{max}} = \frac{1}{2} \left( \frac{v_{\text{max}}}{c} \right)^2
\]

If the fraction change in density is small enough, then it can be neglected:

Criteria 1: \( v_{\text{max}} < < c \)

for air at sea level:

\( c = 342 \text{ m/s} = 700 \text{ mph} \)

for distilled water at 25°C:

\( c = 1500 \text{ m/s} = 3400 \text{ mph} \)

For unsteady flows, a second criteria must be met:

Criteria 2: \( \tau >> \frac{l}{c} \)

where \( \tau = \text{time over which significant changes in } v \text{ occur} \)

\( l = \text{distance over which changes in } v \text{ occur} \)

\( l/c = \text{time for sound to propagate a distance } l \)

For steady flow \( \tau = \infty \) and Criteria 2 is always satisfied.

Any fluid can be considered incompressible if both criteria are met.