Stream Function Relations

2-D Flow

$$\mathbf{v} = \nabla \times [\psi(x,y)\mathbf{e}_z]$$

$$\begin{cases} v_x = \dfrac{\partial \psi}{\partial y} \\ v_y = -\dfrac{\partial \psi}{\partial x} \\ v_z = 0 \end{cases}$$

$$\nabla \times \mathbf{v} = -\left(\nabla^2 \psi\right)\mathbf{e}_z$$

$$\text{curl}^3\, \mathbf{v} = \left[\nabla^2\left(\nabla^2 \psi\right)\right]\mathbf{e}_z$$

Axisymmetric Flow (Cylindrical)

$$\mathbf{v} = \nabla \times \left[\frac{1}{r}\psi(r,z)\,\mathbf{e}_\theta\right]$$

$$\begin{cases} v_r = -\dfrac{1}{r}\dfrac{\partial \psi}{\partial z} \\ v_\theta = 0 \\ v_z = \dfrac{1}{r}\dfrac{\partial \psi}{\partial r} \end{cases}$$

$$\nabla \times \mathbf{v} = -\left(\frac{1}{r}E^2\psi\right)\mathbf{e}_\theta$$

$$\text{curl}^3\, \mathbf{v} = \left[\frac{1}{r}E^2\left(E^2\psi\right)\right]\mathbf{e}_\theta$$

where

$$E^2\psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}$$

Axisymmetric Flow (Spherical)

$$\mathbf{v} = \nabla \times \left[\frac{\psi(r,\theta)}{r\sin\theta}\mathbf{e}_\phi\right]$$

$$\begin{cases} v_r = \dfrac{1}{r^2\sin\theta}\dfrac{\partial \psi}{\partial \theta} \\ v_\theta = -\dfrac{1}{r\sin\theta}\dfrac{\partial \psi}{\partial r} \\ v_\phi = 0 \end{cases}$$

$$\nabla \times \mathbf{v} = -\left(\frac{1}{r\sin\theta}E^2\psi\right)\mathbf{e}_\phi$$

$$\text{curl}^3\, \mathbf{v} = \left[\frac{1}{r\sin\theta}E^2\left(E^2\psi\right)\right]\mathbf{e}_\phi$$

where

$$E^2\psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right)$$