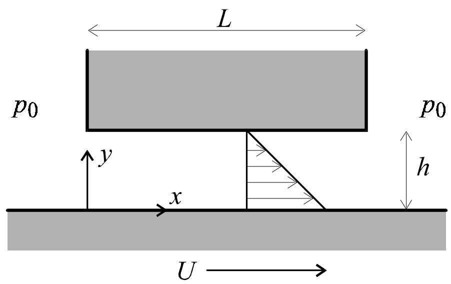
The Lubrication Approximation

Sliding between Parallel Plates

Consider a plate sliding on a lubricating film past a second stationary surface:



If the distance h separating the two plates is small compared to the dimensions of the plate (L and W), we can assume fully developed flow applies through out most of the film. Then:

for $h \ll L$: $$v_x(y) = U\frac{h-y}{h}$$

$$p = p_0 \text{ for all } x,y$$

$$\frac{Q}{W} = \int_0^h v_x(y)dy = \tfrac{1}{2}Uh$$

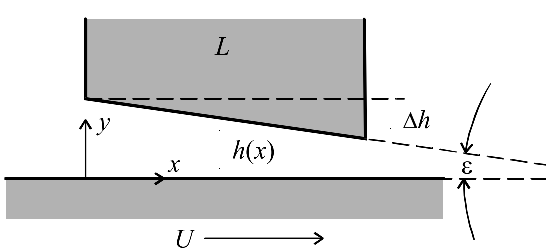
$$\tau_{xy} = \tau_{yx} = -\frac{\mu U}{h}$$

$$F_x = \frac{\mu ULW}{h}$$

Since the pressure is the same inside the film as outside the slide, the sliding motion of two parallel surfaces produces no lateral (y) component of force.

$$F_y = 0$$

Now suppose the slider is inclined ever so slightly relative to the stationary plate.



We might guess, that if ε is small enough, the velocity profile will not be affected. But, owing to the inclination, h is no longer independent of x, so our guess leads to:

$$\frac{Q}{W} = \tfrac{1}{2}Uh(x) = f(x) \qquad (151)$$

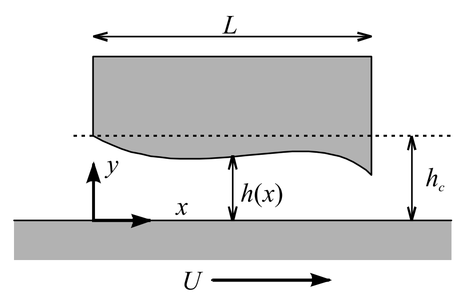
which violates continuity. It turns out that continuity is preserved by a nonzero pressure gradient, dp/dx, which causes pressure-driven flow. Thus even the primary component of the velocity profile is affected by this slight inclination. More significantly, it turns out that this inclination will produce a different pressure in the film from the fluid outside the slider block which, tends to push the two surfaces together or apart.

$$F_y \neq 0$$

Let's try to estimate this force.

General Analysis of Sliding for 2-D Flow

It turns out to be no more difficult to obtain the result for an arbitrary gap profile $h(x)$ (see figure below), since the essential difficulty arises from the fact that h is not constant with respect to x.



Suppose the thickness of the gap is everywhere very small compared to the dimensions of the slider block.

$$h(x) \ll L \quad \text{for all } x$$

Essentially, this is a geometry with two very different length scales characterizing variations in the different directions x and y: we expect slow variations with x and more rapid variations with y. We will exploit this difference using a regular perturbation in the ratio of the two length scales.

We will start by nondimensionalizing the equations of motion:

Let $X \equiv \frac{x}{L}$ $Y \equiv \frac{y}{h_c}$ $u \equiv \frac{v_x}{U}$ $v \equiv \frac{v_y}{v_c}$

L is an obvious choice for the characteristic value of x (since $0<x<L$) and U is a obvious choice for the characteristic value of v_x (since $v_x = U$ at y=0). Since $0<y<h(x)$ some characteristic value h_c of the film thickness seems like a logical choice to scale y.* The choices of characteristic values for v_y and p are not obvious; so we will postpone a choice for now and just denote these values as v_c and p_c.

We seek a solution in the form of a regular perturbation:

where $\alpha \equiv \frac{h_c}{L}$

$$\begin{aligned} v_x(x,y,\alpha) &= Uu(X,Y,\alpha) \\ &= U\left[u_0(X,Y)+\alpha u_1(X,Y)+\ldots\right] \end{aligned} \tag{152}$$

$$\begin{aligned} v_y(x,y,\alpha) &= v_c v(X,Y,\alpha) \\ &= v_c\left[v_0(X,Y)+\alpha v_1(X,Y)+\ldots\right] \end{aligned} \tag{153}$$

$$\begin{aligned} p(x,y,\alpha)-p_\infty &= p_c P(X,Y,\alpha) \\ &= p_c\left[P_0(X,Y)+\alpha P_1(X,Y)+\ldots\right] \end{aligned} \tag{154}$$

As usual, an important aspect of this form is that u and v and all their derivatives with respect to the dimensionless coordinates X and Y are O(α^0):

u,v, $\partial u/\partial X$, $\partial u/\partial Y$, $\partial v/\partial X$ and $\partial v/\partial Y$

The choice of v_c becomes apparent when we nondimensionalize the continuity equation:

continuity: $$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0 \tag{155}$$

Substituting (152) and (153) into (155):

$$\frac{U}{L}\frac{\partial u}{\partial X}=-\frac{v_c}{h_c}\frac{\partial v}{\partial Y}$$

or $$\frac{\partial u}{\partial X}=-\frac{1}{h_c/L}\frac{v_c}{U}\frac{\partial v}{\partial Y}$$

or $$\frac{\partial u}{\partial X}=-\frac{v_c}{\alpha U}\frac{\partial v}{\partial Y}$$

(155) requires the two terms in the continuity equation to be equal but opposite; and to be exactly the same order of α. Since $\partial u/\partial X$ and $\partial v/\partial Y$ are O(α^0), we are forced to choose v_c such that $v_c/\alpha U$ is O(α^0). So let's choose

$$v_c = \alpha U \tag{156}$$

Substituting (152)-(154) and (156) into (155), the leading term is

ε^0: $$\frac{\partial u_0}{\partial X}+\frac{\partial v_0}{\partial Y}=0 \tag{157}$$

Next we examine the principle component of the Navier-Stokes equation:

NSE$_x$: $$\begin{aligned} v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ &+\nu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right) \end{aligned} \tag{158}$$

Substituting (152)-(154) and (156) into (158):

$$\underbrace{\frac{U^2}{L}u\frac{\partial u}{\partial X}}_{O(\alpha^0)}+\underbrace{\frac{\alpha U^2}{\alpha L}v\frac{\partial u}{\partial Y}}_{O(\alpha^0)}=-\underbrace{\frac{p_c}{\rho L}\frac{\partial P}{\partial X}}_{?}+\underbrace{\nu\frac{U}{L^2}\frac{\partial^2 u}{\partial X^2}}_{O(\alpha^0)}$$
$$+\underbrace{\nu\frac{U}{(\alpha L)^2}\frac{\partial^2 u}{\partial Y^2}}_{O(\alpha^{-2})}$$

The last term in the equation is lowest-order in the small parameter α: it's O(α^{-2}). All the other terms in the equation (except possibly for the pressure gradient) are O(α^0). Unless we have some other term in the equation of the same order, we will be forced to take $\partial^2 u/\partial Y^2$ = 0, which yields linear shear flow to all orders in α. We already know that this solution violates macroscopic continuity. To avoid this situation, we choose p_c so that the pressure gradient term is also of O(α^{-2}):

* For definiteness, we might select the largest value of $h(x)$ to be h_c.

$$p_c = \rho L \times \underbrace{\nu}_{\mu/\rho} \frac{U}{(\alpha L)^2} = \frac{\mu U}{\alpha^2 L} \tag{159}$$

With this choice, the leading term in NSE_x is

α^{-2}: $$\frac{\partial^2 u_0}{\partial Y^2} = \frac{\partial P_0}{\partial X} \tag{160}$$

So far we have two equations [(157) and (160)] in three unknows (u_0, v_0 and P_0). We need another equation. So we turn to the secondary component of NSE:

NSE_y: $$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

Introducing our dimensionless variables, we have:

$$\underbrace{\frac{\alpha U^2}{L} u \frac{\partial v}{\partial X}}_{O(\alpha)} + \underbrace{\frac{(\alpha U)^2}{\alpha L} v \frac{\partial v}{\partial Y}}_{O(\alpha)} = -\underbrace{\frac{\mu U}{(\alpha^2 L)(\alpha L)}}_{O(\alpha^{-3})} \frac{\partial P}{\partial Y} + \underbrace{\nu \frac{\alpha U}{L^2} \frac{\partial^2 v}{\partial X^2}}_{O(\alpha)} + \underbrace{\nu \frac{\alpha U}{(\alpha L)^2} \frac{\partial^2 v}{\partial Y^2}}_{O(\alpha^{-1})}$$

After substituting the perturbation expansions (152)-(154), (156) and (159), then collecting terms, the leading term is:

α^{-3}: $$\frac{\partial P_0}{\partial Y} = 0 \quad \text{or} \quad P_0 = \text{const w.r.t. } Y \tag{161}$$

This conclusion suggests that the pressure gradient in (160) can be treated as a constant with respect to Y:

$$\frac{\partial^2 u_0}{\partial Y^2} = \frac{dP_0}{dX} = \text{const w.r.t. } y \tag{162}$$

We might be tempted to set this pressure gradient to zero since:

$$p(0) = p(L) = p_o \tag{163}$$

Of course the pressure gradient is zero when the two plates are parallel. But we already suspect that the pressure gradient is needed to avoid linear shear flow, which violates continuity. So we avoid this temptation and leave dP_0/dX nonzero.

If we are willing to settle for the leading term of the solution, we have no further use for the series or the dimensionless variables. Truncating (152) and (154) after the first term, (162) becomes (in terms of symbols which have units):

$$\frac{\partial^2 v_x}{\partial y^2} = -\frac{1}{\mu}\frac{dp}{dx}$$

Integrating twice:

$$v_x(x,y) = \frac{1}{2\mu}\frac{dp}{dx} y^2 + c_1(x) y + c_2(x)$$

Boundary conditions are given by the "no slip" requirement:

$$v_x = 0 \quad \text{at} \quad y = h(x)$$

$$v_x = U \quad \text{at} \quad y = 0$$

Evaluating the two integration constants leads to:

$$v_x(x,y) = \underbrace{U\left(1 - \frac{y}{h}\right)}_{\text{linear shear flow}} + \underbrace{\frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - yh\right)}_{\text{pressure-driven flow}} \tag{164}$$

The volumetric flowrate per unit width of plate is calculated as:

$$\frac{Q}{W} = \int_0^h v_x(x,y)\,dy = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx} \tag{165}$$

Q can be made to be a constant at each x if dp/dx is allowed to take on non-zero values. The necessary values can calculated from (165):

Q/W=const.: $$\frac{dp}{dx} = \frac{6\mu U}{h^2} - \frac{12\mu}{h^3}\frac{Q}{W} \tag{166}$$

Integrating with respect to x from x=0 where p=p_0 to any other x.

$$p(x) - p_0 = 6\mu U \int_0^x \frac{dx}{h^2} - 12\mu \frac{Q}{W} \int_0^x \frac{dx}{h^3} \tag{167}$$

The pressure is the same at the downstream end of the gap. Then:

$$p(L) - p_0 = 0 = 6\mu U \int_0^L \frac{dx}{h^2} - 12\mu \frac{Q}{W} \int_0^L \frac{dx}{h^3}$$

Knowing the overall pressure drop is zero allows us to compute the flowrate:

$$\frac{Q}{W} = \frac{UH}{2} \tag{168}$$

where $$H = \int_0^L \frac{dx}{h^2} \Big/ \int_0^L \frac{dx}{h^3} \tag{169}$$

is an average gap width. Once H is know we can calculate the pressure gradient by substituting (168) into (166):

$$\frac{dp}{dx} = \frac{6\mu U}{h^2}\left(1 - \frac{H}{h}\right) \tag{170}$$

Special Case: Two Parallel Plates

Let's apply this general result to the case of two parallel plates. Then $h(x)$ = const. and

(169) yields $H = h$

(170) yields $\frac{dp}{dx} = 0$ for all x

so $p(x) = \text{const.} = p_0$

where the constant was determined by the boundary condition (163).

(165) yields $\frac{Q}{W} = \frac{Uh}{2}$

(164) yields $v_x(x,y) = U\left(1 - \frac{y}{h}\right)$

For ε=0, (171) yields

$$F_x = WL\frac{\mu U}{h}$$

and (172) yields $F_y = 0$

All of these results are just as we reported for two parallel plates at the beginning of this chapter. When the slider block is slightly inclined ($|\varepsilon| << 1$ but $\varepsilon \neq 0$), F_x undergoes only a slight perturbation.

Special Case #2: Slightly Nonparallel Plates

If $h(x)$ is a linear function whose value changes from

$$h(0) = h_1$$

to $$h(L) = h_2 < h_1$$

then (169) gives $$H = 2\frac{h_1 h_2}{h_1 + h_2}$$

or $$\frac{1}{H} = \frac{1}{2}\left(\frac{1}{h_1} + \frac{1}{h_2}\right)$$

which is called the "harmonic mean" of h_1 and h_2. Note that

$$h_2 < H < h_1$$

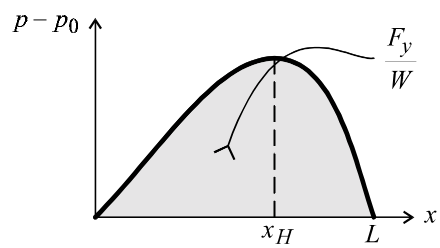
We can evaluate the pressure gradient from (170):

at x=0: $h = h_2 > H \rightarrow dp/dx > 0$

at $x = x_H$: $h = H \rightarrow dp/dx = 0$

at $x = L$: $h = h_1 < H \rightarrow dp/dx < 0$

Thus H represents not only an average gap width (with respect to flowrate) but it is also the width at the point where pressure is a maximum.



Now we are in a position to evaluate the force exerted on the plate by the fluid. The x-component of the force is

$$F_x = W\int_0^L \tau_{yx}(x,0)\,dx = W\int_0^L \mu \left.\frac{dv_x}{dy}\right|_{y=0} dx$$
$$= WL\frac{\mu U}{h} + O(\varepsilon) \tag{171}$$

where $$\varepsilon \equiv \frac{h_1 - h_2}{L}$$

is the angle of tilt between the two plates (see figure on page 100). This result is the same (neglecting the

$O(\varepsilon)$ correction) as would be obtained for two parallel plates. More interesting is the y-component. For a linear gap:

$$\frac{F_y}{W} = \int_0^L \left[p(x) - p_0 \right] dx$$
$$= \frac{6\mu U L^2}{\left(h_1 - h_2\right)^2} \left[\ln\frac{h_1}{h_2} - \frac{2\left(h_1 - h_2\right)}{h_1 + h_2} \right]$$

Does this reduce to zero when the two surfaces are parallel (i.e. $h_1 = h_2$)? Owing to the $(h_1 - h_2)^2$ in the denominator, it appears that $F_y \to \infty$ as $h_1 \to h_2$. But a closer analysis reveals that the terms inside the square brackets tend to zero faster than the denominator outside the brackets. In the limiting case that

$\varepsilon \ll 1$: $\quad h_1 \approx h_2 \approx h,$

and $\quad h_1 - h_2 \ll \min(h_1, h_2),$

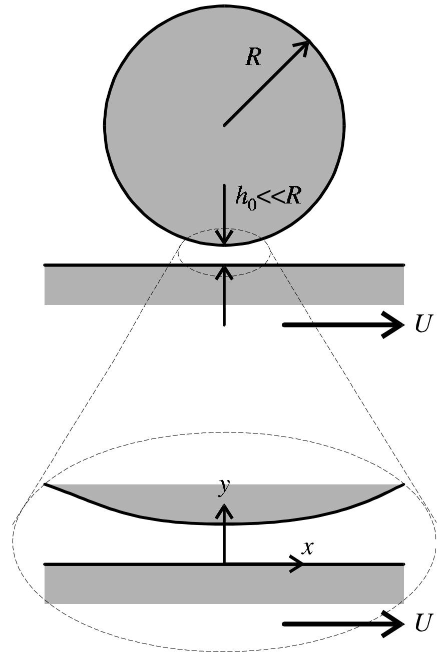
then we obtain:
$$\frac{F_y}{W} = \frac{1}{2}\mu U \left(\frac{L}{h}\right)^3 \varepsilon \tag{172}$$

for the upward force on the slider.

The main difference between two parallel- and two nonparallel-plates is the occurance of the nonzero y-component of lift which vanishes for parallel plates. Notice that $F_y > 0$ if $\varepsilon > 0$ ($h_1 > h_2$) and $F_y < 0$ if $\varepsilon < 0$ ($h_1 < h_2$). Thus either repulsion or attraction of the two plates is possible, depending on the direction of tilt relative to the direction of flow. When the gap is very thin (i.e. $h \ll L$), the magnitude of F_y can be quite significant compared to F_x owing to the h^3 in the denominator of (172).

Translation of a Cylinder Along a Plate

The lubrication approximations developed for the slider block can be easily extended to other geometries. For example, instead of a planar slider block, suppose I try to drag a cylinder parallel to a plate. What will be the force tending to push the two surfaces apart? The same result obtained with the slider block applies here, except we have a different profile $h(x)$ for the gap between the two surfaces.



To deduce the gap profile, recall the equation of a circle

$$\left(x - x_c\right)^2 + \left(y - y_c\right)^2 = R^2$$

where (x_c, y_c) is the location of the center of the circle and R is its radius. Substituting the coordinates of the center in our problem and $y(x) = h(x)$, we have

$$x^2 + \left(h - R - h_0\right)^2 = R^2$$

Dividing by R^2

$$\frac{x^2}{R^2} + \left(\frac{h - h_0}{R} - 1\right)^2 = 1$$

Recognizing that $h - h_0$ is very small compared to R, we can use the Binomial Series, truncated after the second term, to obtain an approximation to the second term:

$$\left(\frac{h - h_0}{R} - 1\right)^2 = \left(1 - \underbrace{\frac{h - h_0}{R}}_{\varepsilon}\right)^2 = (1 - \varepsilon)^2$$
$$= 1 - 2\varepsilon + O\left(\varepsilon^2\right) \approx 1 - 2\frac{h - h_0}{R}$$

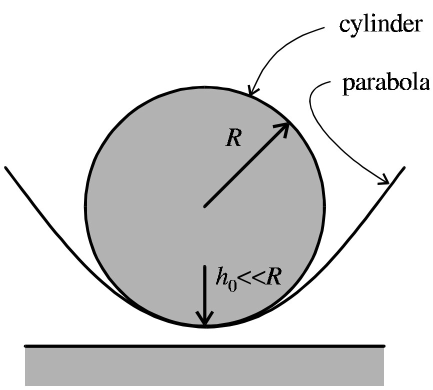
The "1" cancels with the "1" on the right-hand side of the equation, leaving:

$$h(x) = h_0 + \frac{x^2}{2R} \tag{173}$$

provided that $h\text{-}h_0 << R$ which requires that x remain small compared to R.

$$\alpha = \frac{dh}{dx} = \frac{x}{R} << 1$$

where we have moved x=0 to the center of the gap, which is now symmetric about x=0. Although (173) is only valid for x very small, it turns out virtually all of the contribution to the force comes from the region where (173) is valid — provided that h_0 is sufficiently small compared to R. In any case, let's assume that (173) is valid. If that bothers you, then replace the circular cylinder by a parabola.



As with the flat slider, the pressure profile is determined by the need to have the volumetric flowrate through any x=const plane be the same for all such planes. Eq. (165) becomes:

$$\frac{Q}{W} = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx} = \text{const w.r.t. } x \tag{174}$$

If we view Q/W as an unknown integration constant, then we will need two boundary conditions to evaluate the two integration constants we will have after integrating this. Since the fluid held between the cylinder and the wall is in contact with the same reservoir at either end of the gap, we can require:

$$p = p_0 \text{ at } x = -\infty, +\infty$$

By "infinity", we simply mean far from the origin. It might seem more reasonable to specify x=R, but if R is sufficiently large compared to h_0, no significant error will be incurred by extending the limit to infinity. The counterpart to (167) is:

$$p(x) - p_\infty = 6\mu U \int_{-\infty}^{x} \frac{dx'}{h^2} - 12\mu \frac{Q}{W} \int_{-\infty}^{x} \frac{dx'}{h^3} \tag{175}$$

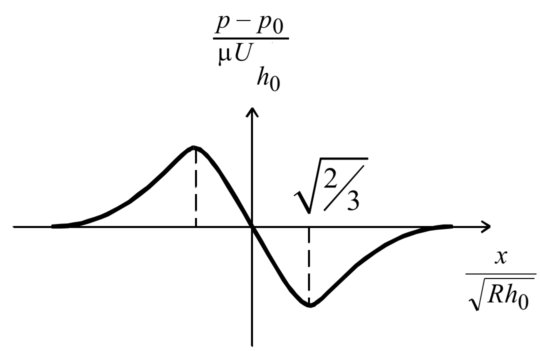
Applying the other boundary condition at x=+∞ allow us to evaluate Q. The counterpart of (168) is

$$\frac{Q}{W} = \frac{UH}{2} \tag{176}$$

where $$H = \int_{-\infty}^{\infty} \frac{dx}{h^2} \Bigg/ \int_{-\infty}^{\infty} \frac{dx}{h^3}$$

is an average gap width.

Substituting (173): $H = \frac{4}{3} h_0$



As long as $h(x)$ is an even function of x, then $p(x)$ must be odd:

$$h(x)\text{=even} \rightarrow p(x)\text{=odd}$$

Thus the pressure profile given by (175) looks as shown at right. The extrema correspond to:

$$dp/dx=0$$

Subsituting (176) and dp/dx=0 into (174) yields:

dp/dx=0: $$h = H = \tfrac{4}{3} h_0$$

Substituting (173) and solving for x:

dp/dx=0: $$x = \pm\sqrt{\tfrac{2}{3} R h_0}$$

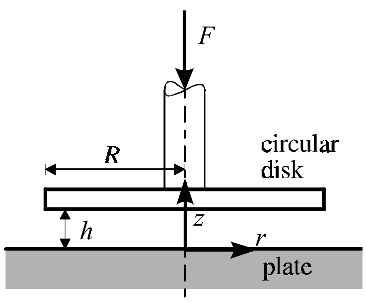
Because p is an odd function, there will be no normal force tending to separate the two surfaces:

p=odd: $$\frac{F_y}{W} = \int_{-\infty}^{\infty} \left[p(x) - p_0 \right] dx = 0$$

$$\frac{F_x}{W} = \int_{-\infty}^{\infty} \tau_{xy}(x)\, dx = 2\pi\mu U \sqrt{\frac{2R}{h_0}}$$

Squeezing Flow

The above two problems both involve the sliding of two surfaces past one another. A related lubrication problem is the squeezing motion between two bodies. Consider the "squeezing" motion in a thin film of liquid held between a circular disk and a parallel flat plate in the limit in which $h \ll R$.



We will adopt a cylindrical coordinate system (r,θ,z) with its origin located on the plate along the axis of the disk (see figure above). If the plate remains stationary, no-slip requires

at z=0: $$v_r = v_\theta = v_z = 0$$

whereas the disk is moving downward:

at z=h: $$v_r = v_\theta = 0 \quad \text{and} \quad v_z = U$$

Owing to axisymmetric nature of the geometry and the boundary conditions, we anticipate that the resulting velocity profile will be axisymmetric (i.e. $v_\theta = 0$ and $\partial/\partial\theta = 0$). In the limit $\varepsilon \equiv h/R \to 0$, a solution to the Navier-Stokes equation can be found via a regular pertubation expansion of the form:

$$v_r(r,z,\varepsilon) = u_c u(\rho,\zeta,\varepsilon) = u_c\left[u_0(\rho,\zeta) + \varepsilon u_1(\rho,\zeta) + \ldots \right]$$

$$v_z(r,z,\varepsilon) = U v(\rho,\zeta,\varepsilon) = U\left[v_0(\rho,\zeta) + \varepsilon v_1(\rho,\zeta) + \ldots \right]$$

$$p(r,z,\varepsilon) - p_\infty = p_c P(\rho,\zeta,\varepsilon) = p_c\left[p_0(\rho,\zeta) + \varepsilon p_1(\rho,\zeta) + \ldots \right]$$

where $$U \equiv \frac{dh}{dt}, \quad \varepsilon \equiv \frac{h}{R}, \quad \rho \equiv r/R \quad \text{and} \quad \zeta \equiv z/h = z/\varepsilon R$$

Note by using the arbitrary u_c as our characteristic radial velocity, we are delaying the choice until we have a chance to inspect the continuity equation:

continuity: $$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \tag{177}$$

Nondimensionalizing:

$$\frac{u_c}{R}\frac{1}{\rho}\frac{\partial(\rho u)}{\partial \rho} + \frac{U}{h}\frac{\partial v}{\partial \zeta} = 0$$

Dividing by U/h:

$$\underbrace{\frac{\varepsilon u_c}{U}}_{1}\underbrace{\frac{1}{\rho}\frac{\partial(\rho u)}{\partial \rho}}_{O(\varepsilon^0)} = \underbrace{-\frac{\partial v}{\partial \zeta}}_{O(\varepsilon^0)} \tag{178}$$

The only way the two sides of (178) can have the same order in ε is for the coefficient to be $O(\varepsilon^0)$. This is accomplished by choosing:

$$u_c = \varepsilon^{-1} U$$

Next we examine the principal component of the Navier-Stokes equation. Although the flow is caused by motion in the z-direction, the r-component is much larger ($u_c \gg U$ for $\varepsilon \ll 1$).

r: $$v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} = -\frac{1}{\rho_f}\frac{\partial p}{\partial r} + \frac{\mu}{\rho_f}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(r v_r)}{\partial r}\right] + \frac{\mu}{\rho_f}\frac{\partial^2 v_r}{\partial z^2}$$

Here we have added the subscript "f" on the fluid density to avoid confusion with the dimensionless radial coordinate. Nondimensionalizing:

$$\underbrace{\frac{\left(\varepsilon^{-1}U\right)^2}{R}u\frac{\partial u}{\partial \rho}}_{O(\varepsilon^{-2})}+\underbrace{\frac{\varepsilon^{-1}U^2}{\varepsilon R}v\frac{\partial u}{\partial \zeta}}_{O(\varepsilon^{-2})}=-\frac{p_c}{\rho_f R}\frac{\partial p}{\partial \rho}$$
$$+\underbrace{\frac{\mu\varepsilon^{-1}U}{\rho_f R^2}\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial(\rho u)}{\partial\rho}\right]}_{O(\varepsilon^{-1})}+\underbrace{\frac{\mu\varepsilon^{-1}U}{\rho_f(\varepsilon R)^2}\frac{\partial^2 u}{\partial\zeta^2}}_{O(\varepsilon^{-3})}$$

Clearly, the first viscous term and both inertial terms are negligible compared to the second viscous term. Unless we have some other term in the equation of the same order, we will be forced to take

$$\partial^2 u / \partial\zeta^2 = 0,$$

which (after no slip is applied) yields $u = 0$ for all ζ. This would violate continuity. Thus we choose p_c so that the pressure derivative has the same order as the second viscous term:

$$\frac{p_c}{\rho_f R}=\frac{\mu\varepsilon^{-1}U}{\rho_f(\varepsilon R)^2}\quad\text{or}\quad p_c=\frac{\mu U}{\varepsilon^3 R}$$

Finally, we look at the secondary component of the Navier-Stokes equation:

$$z:\quad v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{\rho_f}\frac{\partial p}{\partial z}+\frac{\mu}{\rho_f}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{\mu}{\rho_f}\frac{\partial^2 v_z}{\partial z^2}$$

Nondimensionalizing:

$$\underbrace{\frac{\left(\varepsilon^{-1}U\right)U}{R}u\frac{\partial v}{\partial\rho}}_{O(\varepsilon^{-1})}+\underbrace{\frac{U^2}{\varepsilon R}v\frac{\partial v}{\partial\zeta}}_{O(\varepsilon^{-1})}=-\underbrace{\frac{\mu U/\varepsilon^3 R}{\rho_f\varepsilon R}\frac{\partial P}{\partial\zeta}}_{O(\varepsilon^{-4})}$$
$$+\underbrace{\frac{\mu}{\rho_f}\frac{U}{R^2}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v}{\partial\rho}\right)}_{O(\varepsilon^{0})}+\underbrace{\frac{\mu}{\rho_f}\frac{U}{(\varepsilon R)^2}\frac{\partial^2 v}{\partial\zeta^2}}_{O(\varepsilon^{-2})}$$

Note that $\partial P/\partial\zeta$ is the lowest order term in this equation. Moreover, it is the only term which is $O(\varepsilon^{-4})$. Therefore when the perturbation expansions are substituted, and terms of like order are collected, the leading term is

$$\varepsilon^{-4}:\qquad \frac{\partial P_0}{\partial\zeta}=0\quad\text{or}\quad P_0=P_0(\rho)\tag{179}$$

so that the z-dependence of pressure can be neglected compared to the r-dependence. The resulting problem to solve (going back to dimensional quantities) is:

$$\text{Continuity:}\qquad \frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0\tag{180}$$

$$\text{NSE}_r:\qquad \mu\frac{\partial^2 v_r}{\partial z^2}=\frac{dp}{dr}=\text{const. w.r.t. } z\tag{181}$$

Since the right-hand side is independent of z, we can integrate immediately to obtain the general solution:

$$\mu v_r=\frac{dp}{dr}\frac{z^2}{2}+c_1(r)z+c_2(r)$$

Boundary conditions are:

at $z=0$: $v_r = v_z = 0$

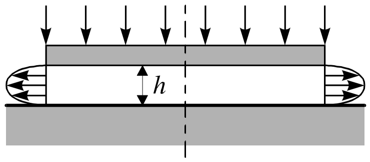
at $z=h$: $v_r = 0$, $v_z = U$

Applying the b.c.'s we get:

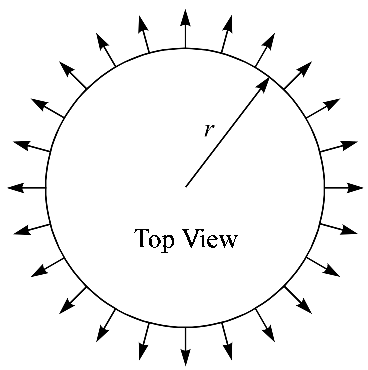
$$v_r=\frac{1}{2\mu}\frac{dp}{dr}z(z-h)\tag{182}$$

As before, dp/dr is determined such that continuity is satisfied. Now, however, macroscopic continuity requires:

$$\underbrace{\int_0^h v_r(r,z)2\pi r\,dz}_{\text{flow out through walls of cylinder}}=\underbrace{-\pi r^2 U}_{\text{flow in through top}}\tag{183}$$



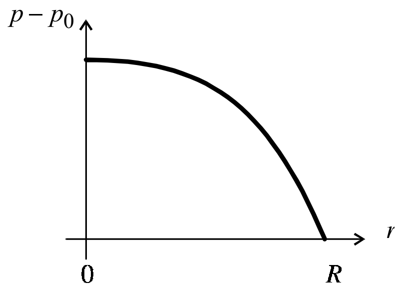
Side View



(182) into (183) and requiring that the result be satisfied for any r yields:

$$\frac{dp}{dr} = 6\mu U \frac{r}{h^3} \tag{184}$$

Requiring that: $p(R) = p_0$



we can integrate to obtain the pressure profile:

$$p(r) - p_0 = 3\mu U \frac{r^2 - R^2}{h^3}$$

which is sketched at right. Substituting (184) into (182):

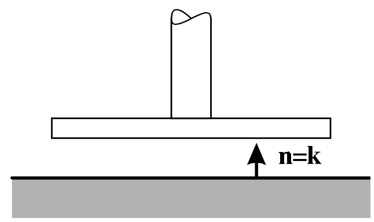
$$v_r(r,z) = -3U\frac{r}{h}\left[\left(\frac{z}{h}\right) - \left(\frac{z}{h}\right)^2\right] \tag{185}$$

The other component of velocity can be determined by applying microscopic continuity. Using (155) and (185):

$$\frac{\partial v_z}{\partial z} = -\frac{1}{r}\frac{\partial(rv_r)}{\partial r} = \frac{6U}{h}\left[\left(\frac{z}{h}\right) - \left(\frac{z}{h}\right)^2\right]$$

Integrating subject to $v_z = 0$ at $z = 0$:

$$v_z(r,z) = U\left[3\left(\frac{z}{h}\right)^2 - 2\left(\frac{z}{h}\right)^3\right] \tag{186}$$



To calculate the force exerted by the plate on the fluid, we use the unit normal pointing out of the fluid: **n**=**k**:

$$d\mathbf{F} = \mathbf{k} \cdot \underline{\underline{\mathbf{T}}}\, da$$

From the axisymmetry of the problem, we anticipate that there will only be a z-component of this force, which we can calculate by post dotting the above by **k**:

$$dF_z = \mathbf{k} \cdot \underline{\underline{\mathbf{T}}} \cdot \mathbf{k}\, da = (-p+\tau_{zz})da$$

In this problem, the normal component of the deviatoric stress (τ_{zz}) vanishes. Using (186):

$$\tau_{zz} = \mu\, \partial v_z/\partial z|_{z=h} = \mu U[6zh^{-2} - 6z^2h^{-3}]|_{z=h} = 0$$

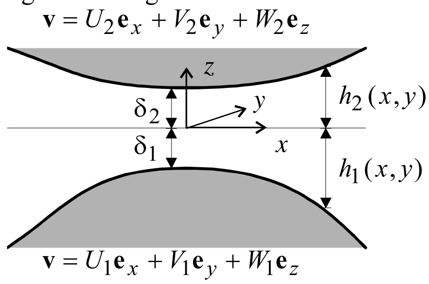
This leaves just a contribution from the pressure. Since $p(r,h)$ is independent of θ, we choose $da = (2\pi r)dr$ to be a thin annulus of radius r and thickness dr:

$$F_z = 2\pi\int_0^R rp(r,h)dr = \frac{3\pi\mu UR^4}{2h^3}$$

Notice that, for a fixed U, the force required becomes unbounded as $h \to 0$.

Reynolds Equation

Sliding motion and squeezing motion are quite different. Yet the lubrication approximation for each has a lot in common. In particular, note that in either case, pressure can be taken as as a constant along a normal to either surface (compare (161) and (179)). In either case, the dominant velocity component is tangent to the surfaces and the principle component of the NSE is approximated by a balance between viscous shear stress on the surface with the pressure gradient along the surface (compare (162) and (181)). It turns out that the lubrication approximation can be generalized to handle an arbitrary combination of squeezing and sliding motion in 3-D.



Consider two bodies of arbitrary (but smooth) shape moving slowly through a viscous fluid in the near vicinity of each other. A rectangular Cartesian coordinate system is chosen so that the *z*-axis coincides with a straight line connecting the two surfaces at the points of minimum approach. The origin is located at some arbitrary point along this line. δ_i represents the distance (along the *z*-axis) from the origin to the surface of body *i* ($i = 1,2$), while $z=-h_1(x,y)$ and $z=h_2(x,y)$ describe a portion of their surfaces nearest the origin.

When the two interacting bodies have simple shapes like a circular cylinder, a sphere or a plate, there will be only one location where they come closest together. When we integrate the pressure and viscous stress over either surface to calculate the force, most of the contribution comes from the region in the throat of the gap where the gap is narrowest. We only need to describe the flow in the vinicity of this throat to calculate the leading term of the forces on the two bodies.

To describe the shape of some arbitrary surfaces like $h_i(x,y)$ in the near vicinity of this throat, we recall the general form for a Taylor series expansion about some arbitrary point (a,b) of some function $f(x,y)$ with two variables x,y:

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!}\left.\frac{\partial^{m+n} f}{\partial x^m \partial y^n}\right|_{a,b} (x-a)^m (y-b)^n$$

$$= f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{a,b}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{a,b}(y-b)$$

$$+\frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{a,b}(x-a)^2 + \frac{2}{2!}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{a,b}(x-a)(y-b)$$

$$+\frac{1}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{a,b}(y-b)^2 + \ldots$$

We are interesting in expanding $h_i(x,y)$ about (0,0) where h_i has a local extremum, which leaves the first partials zero:

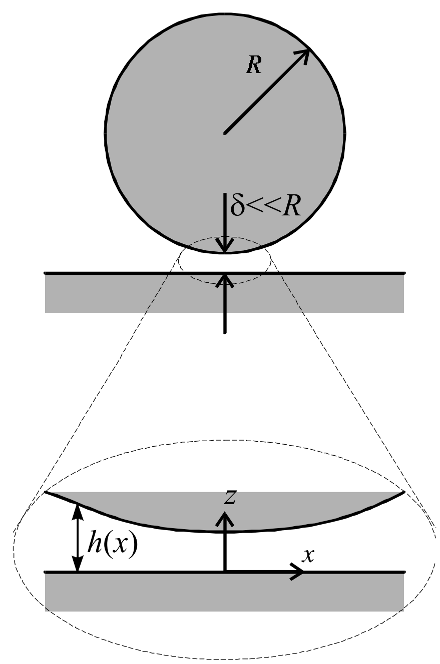
$$h_i(x,y) = \underbrace{h_i(0,0)}_{\delta_i} + \underbrace{\left.\frac{\partial h_i}{\partial x}\right|_{0,0}}_{0} x + \underbrace{\left.\frac{\partial h_i}{\partial y}\right|_{0,0}}_{0} y$$

$$+\underbrace{\frac{1}{2!}\left.\frac{\partial^2 h_i}{\partial x^2}\right|_{0,0}}_{1/R_{ix}} x^2 + \frac{2}{2!}\left.\frac{\partial^2 h_i}{\partial x \partial y}\right|_{0,0} xy \tag{187}$$

$$+\underbrace{\frac{1}{2!}\left.\frac{\partial^2 h_i}{\partial y^2}\right|_{0,0}}_{1/R_{iy}} y^2 + \ldots$$

The second partials are inverses of the two radii of curvature (R_{ix} and R_{iy}) describing any surfaces.

<u>Time out</u>: To see that the coefficients of x^2 and y^2 represent the inverse of radii, consider describing [by $z = h(x)$] the lower surface of a cylinder of radii R which lies parallel to a plate:



Recall the equation of a circle on the xz plane:

$$(x - x_c)^2 + (z - z_c)^2 = R^2$$

where (x_c, z_c) is the location of the center of the circle and R is its radius. In the figure above, the center is located at $x_c=0$ and $z_c = \delta+R$. Substituting the coordinates of the center in our problem and $z = h(x)$ (i.e. the lower surface of the cylinder), we have

$$x^2 + (h - R - \delta)^2 = R^2$$

Dividing by R^2

$$\frac{x^2}{R^2} + \left(\frac{h-\delta}{R} - 1\right)^2 = 1$$

Recognizing that h-h_0 is very small compared to R, we can use the Binomial Series (see page 83), truncated after the second term, to obtain an approximation to the second term:

$$\left(\frac{h-\delta}{R} - 1\right)^2 = \left(1 - \underbrace{\frac{h-\delta}{R}}_{\varepsilon}\right)^2 = (1-\varepsilon)^2$$
$$= 1 - 2\varepsilon + O(\varepsilon^2) \approx 1 - 2\frac{h-\delta}{R}$$

The "1" cancels with the "1" on the right-hand side of the equation, leaving:

$$h(x) = \delta + \frac{x^2}{2R} + O(\varepsilon^2) \tag{188}$$

Notice that the coefficient of the x^2 term is $1/2R$, where R is the radius of the cylinder. Comparing with (187), we see that the second partial is indeed the inverse of the radius of curvature:

$$\left.\frac{d^2h}{dx^2}\right|_0 = \frac{1}{R}$$

<u>Time in</u>. Let R_{ix} and R_{iy} be the radii of curvature of body i in the x- and y-directions, respectively. For distances $h_i(x,y)$ much less than both R_{ix} and R_{iy}, h_i can be approximated by♣

$$h_i(x,y) = \delta_i + \frac{x^2}{2R_{ix}} + \frac{y^2}{2R_{iy}} + O\left(\frac{x^3}{R_{ix}^2}\right)$$

♣ Actually, this assumes that the principle radii of curvature of both surfaces lie either in the x- or y-directions. Should one of the surfaces be rotated around the z-axis by an angle θ, the function acquires an addition term which is proportional to $xy \sin\theta \cos\theta$.

To see how such a form is obtained, read the derivation of (173) starting on page 104. The total distance between the two surfaces is

$$h(x,y) = h_1(x,y) + h_2(x,y) = \delta + \frac{x^2}{2R_x} + \frac{y^2}{2R_y}$$

where $\delta = \delta_1 + \delta_2$ and R_j ($j = x,y$) can be considered to be the radii of curvature of the film:

$$\frac{1}{R_j} = \frac{1}{R_{1j}} + \frac{1}{R_{2j}}$$

As in the two previous examples, scaling leads us to the following equations for the velocity profile in the film:

$$\mu \frac{\partial^2 v_x}{\partial z^2} = \frac{\partial p}{\partial x}$$

$$\mu \frac{\partial^2 v_y}{\partial z^2} = \frac{\partial p}{\partial y}$$

where the pressure profile is independent of z:

$$p = p(x,y)$$

Since pressure is independent of z, these equations can be easily integrated to yield the velocity profile in the film, which again turns out to be the sum of linear shear flow (from the sliding motion) and a parabolic pressure-driven flow.

$$v_x = \underbrace{\frac{\partial p}{\partial x}\frac{1}{2\mu}\left[(z+h_1)(z-h_2)\right]}_{\text{parabolic pressure-driven flow}} + \underbrace{\frac{\Delta U}{h}(z+h_1) + U_1}_{\text{linear shear flow}} \tag{189}$$

$$v_y = \frac{\partial p}{\partial y}\frac{1}{2\mu}\left[(z+h_1)(z-h_2)\right] + \frac{\Delta V}{h}(z+h_1) + V_1$$

where $\Delta U \equiv U_2$-U_1 and $\Delta V \equiv V_2$-V_1

Still unknown is the pressure profile, which is found by requiring the velocity profile to satisfy the continuity equation. In particular, since pressure is independent of z, we will choose p to satisfy the integral of the continuity equation with respect to z

$$\int_{-h_1(x,y,t)}^{h_2(x,y,t)} (\nabla \cdot \mathbf{v})\, dz = 0$$

Expanding the divergence and separating derivatives with respect to z from those with respect to x and y:

$$\int_{-h_1}^{h_2} (\nabla \cdot \mathbf{v})\, dz = \int_{-h_1}^{h_2} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) dz = 0 \qquad (190)$$

We will evaluate this by decomposing the integral of the sum into the sum of the integrals. We will start with the integral of $\partial v_z / \partial z$:

$$\int_{-h_1}^{h_2} \frac{\partial v_z}{\partial z} dz = \int_{-h_1}^{h_2} dv_z = \underbrace{v_z(x,y,h_2)}_{W_2} - \underbrace{v_z(x,y,-h_1)}_{W_1} = \Delta W \qquad (191)$$

For the remaining two integrals, we will interchange the order of integration and differentiation. Applying Leibnitz' rule for differentiating an integral whose limits are functions of the differentiation variable:

$$\frac{\partial}{\partial x} \int_{-h_1(x,y)}^{h_2(x,y)} v_x(x,y,z)\, dz = \int_{-h_1(x,y)}^{h_2(x,y)} \frac{\partial v_x}{\partial x} dz + \underbrace{v_x(x,y,h_2)}_{U_2} \frac{\partial h_2}{\partial x} - \underbrace{v_x(x,y,-h_1)}_{U_1} \frac{\partial(-h_1)}{\partial x}$$

so

$$\int_{-h_1}^{h_2} \frac{\partial v_x}{\partial x} dz = \frac{\partial}{\partial x} \int_{-h_1}^{h_2} v_x(x,y,z)\, dz - U_2 \frac{\partial h_2}{\partial x} - U_1 \frac{\partial h_1}{\partial x} \qquad (192)$$

Substituting the velocity profile (189) into (192) and integrating leads to:

$$\int_{-h_1}^{h_2} v_x\, dz = \frac{h_1 + h_2}{2}(U_1 + U_2) - \frac{(h_1 + h_2)^3}{12\mu} \frac{\partial p}{\partial x}$$

Differentiating

$$\frac{\partial}{\partial x} \int_{-h_1}^{h_2} v_x\, dz = \frac{1}{2}(U_1 + U_2)\left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial x} \right) - \frac{\partial}{\partial x}\left(\frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right)$$

where $h = h_1 + h_2$

(192) becomes

$$\int_{-h_1}^{h_2} \frac{\partial v_x}{\partial x} dz = -\frac{1}{12\mu} \frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x} \right) - \frac{1}{2} \Delta U \frac{\partial \Delta h}{\partial x} \qquad (193)$$

where $\Delta h = h_2 - h_1$

There is a very similar result for the integral of $\partial v_y / \partial y$:

$$\int_{-h_1}^{h_2} \frac{\partial v_y}{\partial y} dz = -\frac{1}{12\mu} \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y} \right) - \frac{1}{2} \Delta V \frac{\partial \Delta h}{\partial y} \qquad (194)$$

Adding (193), (194) and (191), setting to zero (to satisfy (190)) and rearranging:

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y} \right) = -6\mu \Delta U \frac{\partial \Delta h}{\partial x} - 6\mu \Delta V \frac{\partial \Delta h}{\partial y} + 12\mu \Delta W \qquad (195)$$

which is called ***Reynolds lubrication equation***. The solution to this equation yields the pressure profile in the gap for any prescribed translation of the two bodies. Outside the gap, the pressure must approach the bulk pressure, which is taken to be zero

$$p \to 0 \text{ as } (x^2 + y^2) \to \infty$$

In the special case in which upper and lower surfaces are surfaces of revolution around the same axis, polar coordinates (r,θ) are more convenient than (x,y) since then $h_1 = h_1(r)$ and $h_2 = h_2(r)$. (195) can be written in invariant vector notation:

$$\nabla_s \cdot \left(h^3 \nabla_s p \right) = -6\mu (\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{n}_2 - \mathbf{n}_1) \qquad (196)$$

where $\mathbf{v}_i$ is the velocity of body i (i = 1or 2) and $\mathbf{n}_i$ are local normals to body i (not necessarily of unit length). In particular $\mathbf{n}_i$ is defined as:

$$\mathbf{n}_i = \nabla f_i$$

where $f_1(x,y,z) = h_1(x,y) + z$

and $f_2(x,y,z) = h_2(x,y) - z$

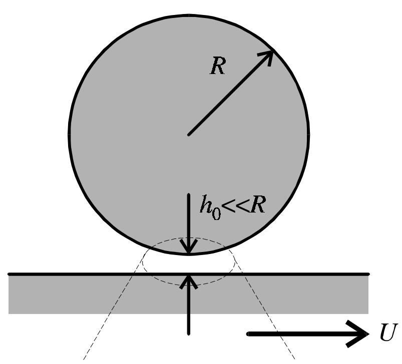
That $\mathbf{n}_i$ are local normals to body i follows from the fact that f_1 is a constant on surface #1 (defined as $z = -h_1$) and f_2 is a constant on surface #2 (defined as $z = +h_2$). If we decompose the velocity of the two bodies into contributions along the z axis and in the xy plane

$$\mathbf{v}_i = \mathbf{v}_{si} + W_i \mathbf{e}_z$$

then (196) becomes

$$\nabla_s \cdot \left(h^3 \nabla_s p\right) = \underbrace{-6\mu\left(\mathbf{v}_{s2} - \mathbf{v}_{s1}\right)\cdot \nabla_s \left(h_2 - h_1\right)}_{\text{sliding motion}} + \underbrace{12\mu \Delta W}_{\text{squeezing motion}} \tag{197}$$

Example #1: Sliding between Cylinder and Plate



Let's reformate the sliding-flow problem for a plate sliding past a cylinder (see page 104). In this problem the equation of the lower surface (the plate) is just

$$h_1(x,y) = 0$$

The equation of the upper surface is

$$h_2(x) = h_0 + \frac{x^2}{2R}$$

The total gap between the two surfaces is

$$h(x) = h_1 + h_2 = h_0 + \frac{x^2}{2R}$$

For Reynolds equation, we also need

$$\Delta h = h_2 - h_1 = h(x)$$

The velocity of the lower surface is

$$U_1 = U \quad V_1 = 0 \quad W_1 = 0$$

while the upper surface is stationary:

$$U_2 = 0 \quad V_2 = 0 \quad W_2 = 0$$

The following quantities appear in Reynolds equation

$$\Delta U = -U \quad \Delta V = 0 \quad \Delta W = 0$$

Reynolds equation becomes

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) = 6\mu U \frac{dh}{dx}$$

which is identical to the derivative of (174). Solving this 2nd order ODE leads to the same pressure profile we determined earlier.

Cavitation

However, there is an important phenomenon which we have not discussed which can cause a lift force to push the two surfaces apart. That phenomenon is

cavitation - formation of gas bubbles caused by a lowering of pressure

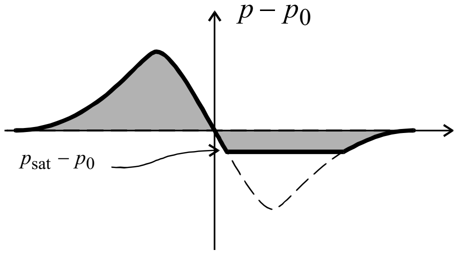
If the absolute pressure of the fluid drops below the vapor pressure of the liquid, we will have boiling of the liquid and cavitation. Because pressures generated in lubrication problems can be significant compared to atmospheric, cavitation is not an uncommon event.

sources of gas bubbles:

- vapor of liquid (if $p<p_{\text{vapor}}$)
- air (if $p<p_{\text{saturation}}$)

Many liquids are kept in contact with air at one atmosphere and therefore become saturated with air. If the pressure on the liquid is suddenly lowered, the air will be supersaturated and air bubbles will form.

What effect will cavitation have on the pressure profile? Although an exact analysis would require consideration of two-phase flow, we can anticipate that — at the very least — the absolute pressure cannot drop below saturation.



If any of the negative portion of the pressure profile is chopped off, the profile loses its anti-symmetry. A repulsive force pushing the surfaces apart becomes likely. The resulting profile might be expected to look something like that shown at right.

Example #2: Squeezing Flow between Disk and Plate

Now let's reformulate the squeezing flow problem on page 106. In this problem the equation of the lower surface (the plate) is just

$$h_1(x,y) = 0$$

The equation of the upper surface is

$$h_2(x,y) = h$$

The total gap between the two surfaces is

$$h = h_1 + h_2 = h$$

For Reynolds equation, we also need

$$\Delta h = h_2 - h_1 = h$$

The velocity of the upper surface is

$$U_2 = 0 \quad V_2 = 0 \quad W_2 = -U$$

while the lower surface is stationary:

$$U_1 = 0 \quad V_1 = 0 \quad W_1 = 0$$

The following quantities appear in Reynolds equation becomes

$$\Delta U = 0 \quad \Delta V = 0 \quad \Delta W = -U$$

(197) becomes

$$\nabla_s \cdot \left(h^3 \nabla_s p\right) = -12\mu U \tag{198}$$

Because the upper surface is a circular disk and the gap is uniform, we expect squeezing flow to be axisymmetric in cylindrical coordinates. In other words, we expect that $p = p(r)$ (i.e. no θ-dependence). In cylindrical (r,θ,z) or polar coordinates (r,θ), the gradient is♣

$$\nabla_s p = \frac{\partial p}{\partial r}\mathbf{e}_r$$

while the divergence is

$$\nabla_s \cdot \left(h^3 \nabla_s p\right) = \frac{1}{r}\frac{d}{dr}\left(rh^3 \frac{dp}{dr}\right)$$

(198) becomes

$$\frac{1}{r}\frac{d}{dr}\left(rh^3 \frac{dp}{dr}\right) = -12\mu U$$

Multiplying through by r and integrating:

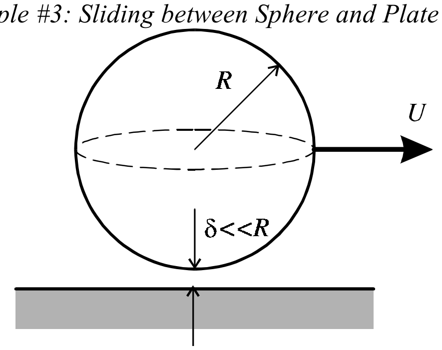
$$rh^3 \frac{dp}{dr} = -6\mu U r^2 + c$$

Dividing by rh^3:

$$\frac{dp}{dr} = -6\mu U \frac{r}{h^3} + \frac{c}{rh^3}$$

When this is integrated a second time, the second term will lead to a logarithmic singularity at r=0. To keep pressure finite, we choose c=0 and then the above equation is identical to (184). Solving this 2nd order ODE leads to the same pressure profile we determined earlier.

Example #3: Sliding between Sphere and Plate



♣ see http://www.andrew.cmu.edu/course/06-703/Vops_cyl.pdf

In this problem the equation of the lower surface (the plate) is just

$$h_1(x,y) = 0$$

The equation of the upper surface is

$$h_2(r) = h_0 + \frac{r^2}{2R}$$

The total gap between the two surfaces is

$$h(r) = h_1 + h_2 = h_0 + \frac{r^2}{2R} \qquad (199)$$

For Reynolds equation, we also need

$$\Delta h = h_2 - h_1 = h(r)$$

The velocity of the sphere is purely along the x-axis

$$\mathbf{v}_2 = \mathbf{v}_{s2} = U\mathbf{e}_x = \mathbf{e}_r\, U\cos\theta - \mathbf{e}_\theta\, U\sin\theta$$

$$U_2 = U \quad V_2 = 0 \quad W_2 = 0$$

while the lower surface is stationary: $\mathbf{v}_1 = \mathbf{v}_{s1} = \mathbf{0}$

The following quantities appear in (197):

$$\nabla_s(h_2 - h_1) = \nabla_s h = \frac{dh}{dr}\mathbf{e}_r$$

$$\begin{aligned} &(\mathbf{v}_{s2} - \mathbf{v}_{s1})\cdot\nabla_s(h_2 - h_1) \\ &= (U\cos\theta\,\mathbf{e}_r - U\sin\theta\,\mathbf{e}_\theta)\cdot\left(\frac{dh}{dr}\mathbf{e}_r\right) \\ &= U\frac{dh}{dr}\cos\theta \end{aligned}$$

(197) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial p}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{h^3}{r}\frac{\partial p}{\partial\theta}\right) = -6\mu U\frac{dh}{dr}\cos\theta$$

For the particular case in which h is given by (199), the solution is

$$p(r,\theta) = p_\infty + \frac{6\mu U r\cos\theta}{5h^2}$$

This produces no force in the z-direction (pressure profile is antisymmetric) but of course a force must be applied to the sphere to get it to move:

$$F_x = \frac{16}{5}\pi\mu UR\ln\frac{R}{\delta} + O(\delta^0) \qquad \text{as } \delta\to 0$$

$$F_y = F_z = 0$$

