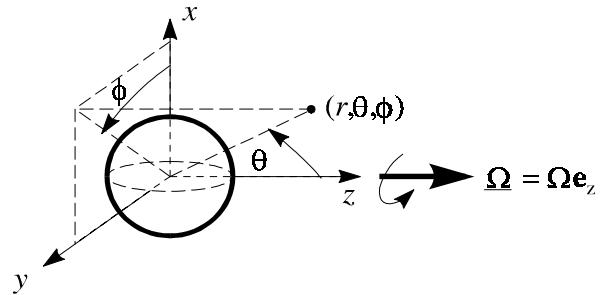


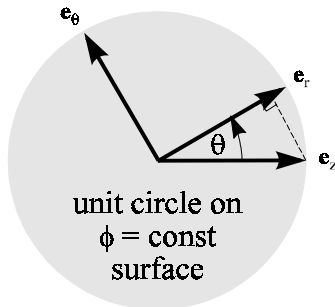
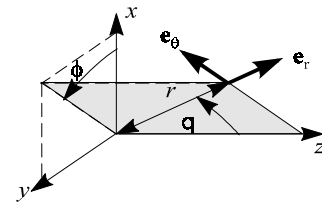
## Key to Homework #7

1a.) First, we recall the definition of the spherical coordinates relative to Cartesian coordinates (see figure at right). We choose to rotate the sphere about the z-axis (in the  $\phi$ -direction). The sphere itself is undergoing rigid-body rotation:



$$\mathbf{v} = \underline{\underline{\Omega}} \times \mathbf{r}$$

where the vector  $\underline{\underline{\Omega}}$  is aligned with the z-axis. To express this velocity profile in component form in spherical coordinates, we need to express the unit vector  $\mathbf{e}_z$  in terms of the unit vectors in spherical coordinates. Referring to the figure at right, we note that  $\mathbf{e}_r$ ,  $\mathbf{e}_\theta$ , and  $\mathbf{e}_z$  all lie in the same  $\phi = \text{const}$  plane.



If we shift all three unit vectors to the origin (recall that the origin is not part of the definition of any vector) and re-orient the  $\phi = \text{const}$  plane to coincide with the plane of the page, then we get

$$\mathbf{e}_z = (\cos\theta)\mathbf{e}_r + (\sin\theta)(-\mathbf{e}_\theta) \quad (1)$$

$$\mathbf{v} = \underline{\underline{\Omega}} \times \mathbf{r} = (\Omega \mathbf{e}_z) \times (r \mathbf{e}_r)$$

Substituting for  $\mathbf{e}_z$ , we get

$$\mathbf{v} = \Omega [(\cos\theta)\mathbf{e}_r - (\sin\theta)\mathbf{e}_\theta] \times (r \mathbf{e}_r) = r\Omega \cos\theta (\mathbf{e}_r \times \mathbf{e}_r) - r\Omega \sin\theta (\mathbf{e}_\theta \times \mathbf{e}_r) = r\Omega (\sin\theta) \mathbf{e}_\phi$$

for rigid-body rotation. Applying the “no-slip” condition at the surface of the sphere, we obtain one of the boundary conditions:

$$\text{at } r=R: \quad v_r = v_\theta = 0, \quad v_\phi = R\Omega \sin\theta \quad (2)$$

The second boundary condition is obtained by recognizing that, far from the rotating sphere, the fluid remains undisturbed:

$$\text{as } r \rightarrow \infty: \quad v_r = v_\theta = v_\phi = 0 \quad (3)$$

Based on the form of these boundary conditions, we guess the solution has the form:

$$v_r = v_\theta = 0, \quad v_\phi = f(r) \sin\theta \quad (4)$$

and that

$$\partial p / \partial \phi = 0$$

This form automatically satisfies continuity for an incompressible fluid:

$$\nabla \cdot \mathbf{v} = \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} = 0$$

The remaining terms in the Navier-Stokes equation (Whitaker, p156) are:

$$r: \quad -\rho \frac{v_\phi^2}{r} = -\frac{\partial p}{\partial r}$$

$$\theta: \quad -\rho \frac{v_\phi^2}{r} \cot \theta = -\frac{1}{r} \frac{\partial p}{\partial \theta}$$

$$\phi: \quad 0 = \mu \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial v_\phi}{\partial r} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial v_\phi}{\partial \theta} \right] - \frac{v_\phi}{r^2 \sin^2 \theta} \right\} \quad (5)$$

$$\text{Substitute (4) into (5):} \quad \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} - \frac{2}{r^2} f = 0 \quad (6)$$

after the  $\sin \theta$  was cancelled from each term. Substitute (4) into (3) and (2):

$$\text{at } r=R: \quad f = R\Omega \quad (7)$$

$$\text{as } r \rightarrow \infty: \quad f = 0 \quad (8)$$

(6) is a Cauchy-Euler equation with solutions of the form  $f = r^n$ . Substituting this form into (6), we obtain

$$\begin{aligned} n(n-1)r^{n-2} + 2nr^{n-2} - 2r^{n-2} &= 0 \\ (n-1)(n+2)r^{n-2} &= 0 \end{aligned} \quad (9)$$

The roots of this characteristic polynomial are  $n = -2$  and  $1$ . The general solution of (6) is

$$f(r) = Ar + \frac{B}{r^2} \quad (10)$$

To satisfy boundary conditions (7) and (8), we must choose  $A=0$  and  $B=R^3\Omega$ . Then equation (4) becomes

$$v_\phi(r, \theta) = \frac{R^3 \Omega \sin \theta}{r^2} \quad (11)$$

1b.) Generally the torque  $\underline{\mathbf{T}}$  applied to a body whose outward pointing normal is  $\mathbf{n}$  can be computed from

$$\underline{\mathbf{T}} = \oint_{\mathbb{A}} \mathbf{r} \times (\mathbf{n} \cdot \underline{\mathbf{T}}) da \quad (12)$$

We wish to compute the torque which must be applied externally to rotate the sphere at a steady angular velocity  $\underline{\Omega}$ . This external torque will be transmitted by the sphere to the fluid and is vectorially the same as the torque the sphere exerts on the fluid. Thus we choose  $\mathbb{A}$  to be the surface  $r=R$  of the sphere. Since the body we are computing the torque *on* is the fluid, we choose  $\mathbf{n} = -\mathbf{e}_r$  as the *outward* pointing normal:

$$\mathbf{n} \cdot \underline{\mathbf{T}} = (-\mathbf{e}_r) \cdot \left( \sum_i \sum_j T_{ij} \mathbf{e}_i \mathbf{e}_j \right) = -T_{rr} \mathbf{e}_r - T_{r\theta} \mathbf{e}_\theta - T_{r\phi} \mathbf{e}_\phi \quad (13)$$

The lever arm drawn from the center to any point on the surface of the sphere is  $\mathbf{r} = R\mathbf{e}_r$ , so

$$\begin{aligned} \mathbf{r} \times (\mathbf{n} \cdot \underline{\mathbf{T}}) &= (R\mathbf{e}_r) \times (-T_{rr} \mathbf{e}_r - T_{r\theta} \mathbf{e}_\theta - T_{r\phi} \mathbf{e}_\phi) \\ &= -RT_{rr} (\mathbf{e}_r \times \mathbf{e}_r) - RT_{r\theta} (\mathbf{e}_r \times \mathbf{e}_\theta) - RT_{r\phi} (\mathbf{e}_r \times \mathbf{e}_\phi) \\ &= RT_{rr} (\mathbf{0}) - RT_{r\theta} (\mathbf{e}_\phi) - RT_{r\phi} (-\mathbf{e}_\theta) \\ &= RT_{r\phi} \mathbf{e}_\theta - RT_{r\theta} \mathbf{e}_\phi \end{aligned} \quad (14)$$

We anticipate that the net torque vector will be parallel to angular velocity vector, which in turn is parallel to  $\mathbf{e}_z$ . So we will focus our attention on computing the z-component of the torque:

$$\mathbf{e}_z \cdot [\mathbf{r} \times (\mathbf{n} \cdot \underline{\mathbf{T}})] = [(\cos\theta)\mathbf{e}_r - (\sin\theta)\mathbf{e}_\theta] \cdot (RT_{r\phi} \mathbf{e}_\theta - RT_{r\theta} \mathbf{e}_\phi) = -R(\sin\theta)T_{r\phi} \quad (15)$$

The z-component of (12) becomes

$$\mathbf{e}_z \cdot \underline{\mathbf{T}} = \mathbf{e}_z \cdot \oint_{\mathbb{A}} \mathbf{r} \times (\mathbf{n} \cdot \underline{\mathbf{T}}) da = \oint_{\mathbb{A}} \mathbf{e}_z \cdot [\mathbf{r} \times (\mathbf{n} \cdot \underline{\mathbf{T}})] da = -R \oint_{\mathbb{A}} (\sin\theta) T_{r\phi} da \quad (16)$$

The stress is related to the velocity profile by equation (f) in Table 5.2-4 on p146 of Whitaker:

$$T_{r\phi} = \tau_{r\phi} = \mu r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \quad (17)$$

Substituting (11)

$$T_{r\phi} = \mu r R^3 \Omega \sin \theta \frac{\partial}{\partial r} (r^{-3}) = -3\mu r^{-3} R^3 \Omega \sin \theta$$

$$T_{r\phi} \Big|_{r=R} = -3\mu \Omega \sin \theta \quad (18)$$

Since the integrand depends only on  $\theta$ , we choose  $da$  to be  $2\pi(R\sin\theta)(Rd\theta)$ . Substituting (18) into (16) yields

$$\begin{aligned} \mathbb{T}_z &= -R \int_0^\pi (\sin \theta) (-3\mu \Omega \sin \theta) (2\pi R \sin \theta) (R d\theta) \\ &= 6\pi \mu \Omega R^3 \underbrace{\int_0^\pi \sin^3 \theta d\theta}_{4/3} = 8\pi \mu \Omega R^3 \end{aligned} \quad (19)$$

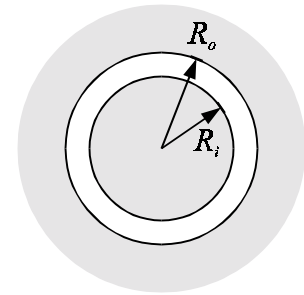
Multiplying both sides by  $\mathbf{e}_z$ , we have

$$\underline{\mathbf{T}} = 8\pi \mu R^3 \underline{\Omega} \quad (20)$$

2a.) In cylindrical coordinates, the “no-slip” boundary conditions are:

$$\begin{aligned} \text{at } r=R_i: \quad & v_\theta = R_i \Omega_i \\ & v_r = v_z = 0 \end{aligned}$$

$$\begin{aligned} \text{at } r=R_o: \quad & v_\theta = R_o \Omega_o \\ & v_r = v_z = 0 \end{aligned}$$



Based on these b.c.'s we expect the velocity profile to generally have the following form:

$$\begin{aligned} v_\theta &= v_\theta(r) \\ v_r &= v_z = 0 \end{aligned}$$

This automatically satisfies continuity. Neglecting gravity, the Navier-Stokes equations in cylindrical coordinates are (see BSL p85):

$$\text{NSE}_r: \quad \underbrace{-\rho \frac{v_\theta^2}{r}}_{\text{inertial}} = -\frac{\partial p}{\partial r}$$

$$\text{NSE}_\theta: \quad 0 = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r v_\theta) \right)$$

$$\text{NSE}_z: \quad 0 = -\frac{\partial p}{\partial z}$$

Based on  $NSE_r$  and  $NSE_z$  (neglecting “inertial terms”), we conclude that pressure can depend only on  $\theta$ , but  $NSE_\theta$  yields

$$\underbrace{\mu r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (rv_\theta) \right)}_{f(r)} = \underbrace{\frac{dp}{d\theta}}_{g(\theta)} = \text{const} = 0$$

Clearly,  $dp/d\theta$  must be a constant, but the only constant which also leads to a periodic function of  $\theta$  is zero.\* We have an exact differential, which is easy to integrate:

$$\frac{1}{r} \frac{d}{dr} (rv_\theta) = c_1$$

$$\frac{d}{dr} (rv_\theta) = c_1 r$$

$$rv_\theta = \frac{1}{2} c_1 r^2 + c_2$$

$$v_\theta = \frac{1}{2} c_1 r + \frac{c_2}{r}$$

Choosing the two integration constants to satisfy the b.c.’s, the particular solution becomes

$$v_\theta(r) = \frac{R_i^2 \Omega_i - R_o^2 \Omega_o}{R_i^2 - R_o^2} r + \frac{R_i^2 R_o^2 (\Omega_o - \Omega_i)}{R_i^2 - R_o^2} \frac{1}{r}$$

2b.) If the outer cylinder remains stationary,  $\Omega_o = 0$  and the velocity profile becomes

$$v_\theta(r) = \frac{R_i^2 \Omega_i}{R_i^2 - R_o^2} \left( r - \frac{R_o^2}{r} \right) \quad (1)$$

As explained on p98 of the Notes, the torque exerted by the fluid on any surface  $\mathbf{n}$  points out of is given by

$$\underline{\mathbf{T}} = \int_A \mathbf{r} \times \underbrace{(\mathbf{n} \cdot \underline{\mathbf{T}})}_{d\mathbf{F}}$$

Let’s evaluate the torque on the inner cylinder, whose outward normal is  $\mathbf{n} = \mathbf{e}_r$ :

\* Even if we did not neglect inertial terms in  $NSE_r$ , we would not expect pressure to depend on  $\theta$ , owing to axial symmetry of two concentric circular cylinders.

$$\mathbf{n} \cdot \underline{\underline{\mathbf{T}}} = \mathbf{e}_r \cdot \underline{\underline{\mathbf{T}}} = T_{rr} \mathbf{e}_r + T_{r\theta} \mathbf{e}_\theta + \underbrace{T_{rz}}_0 \mathbf{e}_z$$

In cylindrical coordinates, the position vector is  $\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z$ . Pre-crossing by this vector yields

$$\mathbf{r} \times (\mathbf{n} \cdot \underline{\underline{\mathbf{T}}}) = (r\mathbf{e}_r + z\mathbf{e}_z) \times (T_{rr} \mathbf{e}_r + T_{r\theta} \mathbf{e}_\theta) = r \underbrace{T_{r\theta}}_{\tau_{r\theta}} \mathbf{e}_z + zT_{rr} \mathbf{e}_\theta - zT_{r\theta} \mathbf{e}_r$$

Anticipating that the main torque will act in the  $z$ -direction, we pre-dot both sides by  $\mathbf{e}_z$ :

$$\mathcal{T}_z = \int_A r \tau_{r\theta} da \quad (2)$$

Looking up  $\tau_{r\theta}$  in BSL (p89, after changing the sign):

$$\tau_{r\theta} = \mu r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right)$$

Substituting (1):

$$\tau_{r\theta} = \mu r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) = \frac{\mu R_i^2 \Omega_i}{R_i^2 - R_o^2} r \frac{\partial}{\partial r} (1 - R_o^2 r^{-2}) = \frac{\mu R_i^2 \Omega_i}{R_i^2 - R_o^2} r (2R_o^2 r^{-3}) = \frac{2\mu R_i^2 R_o^2 \Omega_i}{R_i^2 - R_o^2} r^{-2}$$

Evaluating at  $r = R_i$ :

$$\tau_{r\theta}|_{r=R_i} = \frac{2\mu R_o^2 \Omega_i}{R_i^2 - R_o^2} \quad (3)$$

The  $A$  in (2) corresponds to  $r = R_i = \text{const}$ . Thus the entire integrand of (2) is a constant and can be factored out, leaving

$$\mathcal{T}_z = R_i \tau_{r\theta} \int da = 4\pi R_i^2 L \tau_{r\theta}$$

$\underbrace{A}_{2\pi R_i L}$

Substituting (3):

$$\frac{\mathcal{T}_z}{L} = 4\pi\mu\Omega_i \frac{R_i^2 R_o^2}{R_i^2 - R_o^2}$$

This is the torque exerted by the fluid. The externally applied torque is opposite in sign:

inner cylinder:

$$\frac{\mathcal{T}_z}{L} = 4\pi\mu\Omega_i \frac{R_i^2 R_o^2}{R_o^2 - R_i^2}$$

The torque applied to the outer cylinder must be equal in magnitude but opposite in sign (since the net externally applied torque is zero).

3.) The problem is to solve (see Notes, p104):

$$E^2(E^2\psi)=0: \quad g''-2r^2g=0 \quad (4)$$

$$\text{where } g \text{ is related to } f \text{ by:} \quad f''-2r^2f=g(r) \quad (5)$$

Substituting (5) into (4), expanding the derivative and multiplying through by  $r^4$ :

$$r^4 \frac{d^4 f}{dr^4} - 4r^2 \frac{d^2 f}{dr^2} + 8r \frac{df}{dr} - 8f = 0$$

which we recognize as a Cauchy-Euler equation. Looking for solutions of the form  $f = r^n$ , we obtain the polynomial:

$$\begin{aligned} n(n-1)(n-2)(n-3) - 4n(n-1) + 8n - 8 &= 0 \\ n(n-1)(n-2)(n-3) - 4n(n-1) + 8(n-1) &= 0 \\ (n-1)[n(n-2)(n-3) - 4n + 8] &= 0 \\ (n-1)[n(n-2)(n-3) - 4(n-2)] &= 0 \\ (n-1)(n-2)[n(n-3) - 4] &= 0 \\ (n-1)(n-2)(n^2 - 3n - 4) &= 0 \\ (n-1)(n-2)(n-4)(n+1) &= 0 \end{aligned}$$

which has four distinct roots:  $n = -1, 1, 2, \text{ and } 4$

from which we can construct the general solution:

$$f(r) = c_1 r^{-1} + c_2 r + c_3 r^2 + c_4 r^4 \quad (6)$$

The corresponding general solution for the streamfunction is

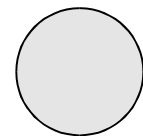
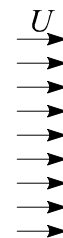
$$\boxed{\psi(r, \theta) = (c_1 r^{-1} + c_2 r + c_3 r^2 + c_4 r^4) \sin^2 \theta} \quad (7)$$

4a.) The problem is to solve

$$\text{curl}^3 \mathbf{v} = \mathbf{0}$$

for 2-D flow, using a streamfunction. On page 59 of Notes, we saw that the relationship between streamfunction and velocity involves the curl:

$$\mathbf{v} = \nabla \times [\psi \mathbf{e}_z] = \underbrace{\frac{1}{r} \frac{\partial \psi}{\partial \theta}}_{v_r} \mathbf{e}_r - \underbrace{\frac{\partial \psi}{\partial r}}_{v_\theta} \mathbf{e}_\theta \quad (8)$$



end view of cylinder

For 2-D flow in class, we employed Cartesian coordinates; for this problem cylindrical coordinates are more convenient (again the z-axis coincides with the axis of the cylinder). The second equation above was obtained by expanding the curl in cylindrical coordinates ([http://www.andrew.cmu.edu/course/06-703/Vops\\_cyl.pdf](http://www.andrew.cmu.edu/course/06-703/Vops_cyl.pdf))

Taking the curl: 
$$\nabla \times \mathbf{v} = \text{curl}^2(\psi \mathbf{e}_z) = -(\nabla^2 \psi) \mathbf{e}_z \quad (9)$$

If we take the curl twice more:

$$\text{curl}^3 \mathbf{v} = \text{curl}^4(\psi \mathbf{e}_z) = \text{curl}^2[-(\nabla^2 \psi) \mathbf{e}_z]$$

The argument of this  $\text{curl}^2$  has the same form as the  $\text{curl}^2$  in (9), so we can use (9) to obtain the result

$$\text{curl}^3 \mathbf{v} = -\nabla^2[-(\nabla^2 \psi) \mathbf{e}_z] = \nabla^2(\nabla^2 \psi) \mathbf{e}_z$$

So we choose  $\psi(r, \theta)$  to satisfy 
$$\nabla^2(\nabla^2 \psi) = 0 \quad (10)$$

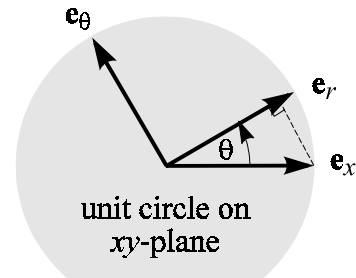
where 
$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}$$

Far from the cylinder, the flow becomes uniform:

as  $r \rightarrow \infty$ : 
$$\mathbf{v} \rightarrow U \mathbf{e}_x$$

We can use the geometry of the right triangle in the figure at right to translate this expression into cylindrical coordinates:

as  $r \rightarrow \infty$ : 
$$\mathbf{v} \rightarrow U[(\cos \theta) \mathbf{e}_r - (\sin \theta) \mathbf{e}_\theta]$$



Matching corresponding components of this boundary condition and (8):

as  $r \rightarrow \infty$ : 
$$\begin{cases} v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = U \cos \theta \\ v_\theta = -\frac{\partial \psi}{\partial r} = -U \sin \theta \end{cases}$$

Integrating either one of these two PDE's yields:

as  $r \rightarrow \infty$ : 
$$\psi \rightarrow Ur \sin \theta \quad (11)$$

The second boundary conditions is “no slip” on the surface of the cylinder:

at  $r=R$ : 
$$\frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial r} = 0 \quad (12)$$

So the mathematical problem which remains is (10) subject to (11) and (12). Based on the form of (11), we seek a solution having the form

$$\psi(r, \theta) = f(r) \sin \theta \quad (13)$$

4b.) Taking the Laplacian in cylindrical coordinates (see either p131 or p739 of BS&L):

$$\nabla^2 \psi = \left( \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{f}{r^2} \right) \sin \theta$$

$$\nabla^2 (\nabla^2 \psi) = \left( r^4 \frac{d^4 f}{dr^4} + 2r^3 \frac{d^3 f}{dr^3} - 3r^2 \frac{d^2 f}{dr^2} + 3r \frac{df}{dr} - 3f \right) \frac{\sin \theta}{r^4} = 0$$

Multiplying by  $r^4$  and dividing by  $\sin \theta$ :

$$r^4 \frac{d^4 f}{dr^4} + 2r^3 \frac{d^3 f}{dr^3} - 3r^2 \frac{d^2 f}{dr^2} + 3r \frac{df}{dr} - 3f = 0 \quad (14)$$

which is another Cauchy-Euler equation. Looking for solutions of the form  $f = r^n$  generates the polynomial:

$$n^4 - 4n^3 + 2n^2 + 4n - 3 = 0$$

which can be factored as  $(n+1)(n-1)^2(n-3) = 0$

This polynomial has only three distinct roots:  $n = -1, +1, \text{ and } +3$ . One of the roots has multiplicity of two. Thus this method yields only three of the four linearly independent solutions of the fourth-order ODE. The fourth solution turns out to be  $r \ln r$  which can be found by variation of parameters. The general solution becomes

$$f(r) = c_1 r^{-1} + c_2 r + c_3 r^3 + c_4 r \ln r$$

Boundary conditions are

(11) becomes: 
$$f(r) \rightarrow Ur \quad \text{as } r \rightarrow \infty \quad (15)$$

(12) becomes: 
$$f(R) = f'(R) = 0 \quad (16)$$

Since both  $r^3$  and  $r \ln r$  blow-up faster than  $r$ ,\* we must choose their coefficients to vanish

\*  $r \ln r$  behaves like  $r^{1+\epsilon}$  as  $r \rightarrow \infty$ , where  $\epsilon$  is an arbitrarily small positive number.

in order to satisfy (15):

$$c_3 = c_4 = 0 \quad \text{and} \quad c_2 = U$$

This leaves only one remaining integration constant to satisfy both conditions in (16). Different values of  $c_1$  are required for each condition. Thus no choice of the integration constants can satisfy all of the conditions.

5a.) This is a continuation of problem for axisymmetric creeping flow around a sphere. Recalling the form of the solution

$$\psi(r, \theta) = f(r) \sin^2 \theta \quad (17)$$

the boundary conditions on  $\psi$  translate into the following conditions on  $f$ :

$$r \rightarrow \infty: \quad f \rightarrow (1/2)Ur^2$$

$$r=R: \quad f = df/dr = 0$$

The general solution to Stokes equation for axisymmetric flow around a sphere was found in Hwk #7 to be

$$f(r) = c_1 r^{-1} + c_2 r + c_3 r^2 + c_4 r^4 \quad (18)$$

To get (18) to behave like  $r^2$  as  $r \rightarrow \infty$ , we must kill the  $r^4$  term which will otherwise dominate, by choosing  $c_4 = 0$ . To get the coefficient of  $r^2$  to match the b.c., we choose  $c_3 = U/2$ . (18) becomes

$$f(r) = c_1 r^{-1} + c_2 r + \frac{1}{2}Ur^2 \quad (19)$$

The two remaining b.c.'s give

$$v_r=0 \text{ at } r=R: \quad f(R) = \frac{c_1}{R} + c_2 R + \frac{1}{2}UR^2 = 0 \quad (20)$$

$$v_\theta=0 \text{ at } r=R: \quad f'(R) = -\frac{c_1}{R^2} + c_2 + UR = 0$$

which can be readily solved to obtain  $c_1$  and  $c_2$ . The final results after substituting back into (18) is

$$\psi(r, \theta) = UR^2 \left[ \frac{1}{4} \frac{R}{r} - \frac{3}{4} \frac{r}{R} + \frac{1}{2} \left( \frac{r}{R} \right)^2 \right] \sin^2 \theta$$

5b.) The boundary conditions are the same as in the last case, except that  $v_\theta = 0$  at  $r = R$  is not imposed. After the b.c. as  $r \rightarrow \infty$  is imposed, we have (19):

$$f^o(r) = c_1^o r^{-1} + c_2^o r + \frac{1}{2} U r^2 \quad (19)$$

In addition, we impose (20):

$$v_r=0 \text{ at } r=R: \quad f^o(R) = \frac{c_1^o}{R} + c_2^o R + \frac{1}{2} U R^2 = 0 \quad (20)$$

After solving for  $c_2$  in terms of  $c_1$ , (19) becomes:

$$f^o(r) = \frac{c_1^o}{r} + \left( -\frac{1}{2} U R - \frac{c_1^o}{R^2} \right) r + \frac{1}{2} U r^2 \quad (21)$$

In preparation for solving part d), we will need to evaluate  $\tau_{r\theta}(R, \theta)$ . When Newton's law of viscosity holds, shear stress is related to the velocity profile by the tables in BSL or Whitaker. For cylindrical coordinates, we use BSL tables on p89 (changing the sign to adopt Whitaker's convention):

$$\tau_{r\theta} = \mu \left[ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (22)$$

where velocity is related to streamfunction by

$$v_r = \frac{1}{r^2} \frac{\partial \psi}{\sin \theta} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \quad (23)$$

After substituting (17) into (23) and the result into (22) we obtain:

$$\tau_{r\theta} = -\mu \left( r^2 \frac{d^2 f}{dr^2} - 2r \frac{df}{dr} + 2f \right) \frac{\sin \theta}{r^3} \quad (24)$$

Substituting (21)

$$\tau_{r\theta}^o(r, \theta) = -6\mu^o c_1^o \frac{\sin \theta}{r^4} \quad (25)$$

5c.) Let's first look at the b.c. at  $r = 0$ . Substituting (17) into (23):

$$v_r = \frac{2f(r)}{r^2} \cos \theta \quad \text{and} \quad v_\theta = -\frac{1}{r} \frac{df}{dr} \sin \theta \quad (26)$$

Unless  $f(r) \rightarrow 0$  as  $r \rightarrow 0$  as least as fast as  $r^2 \rightarrow 0$ , both velocity components will be unbounded. Thus we must choose  $c_1 = c_2 = 0$  in (18), leaving

for  $r < R$ : 
$$f^i(r) = c_3^i r^2 + c_4^i r^4$$

To get  $v_r$  to vanish at  $r=R$ , we must choose  $c_3 = -c_4 R^2$ , leaving

for  $r < R$ : 
$$f^i(r) = c_4^i (-R^2 r^2 + r^4) \quad (27)$$

In preparation for solving part d), we will need to evaluate  $\tau_{r\theta}(R, \theta)$ . Substituting this result into (24):

$$\tau_{r\theta}^i = -6\mu^i c_4^i r \sin\theta \quad (28)$$

5d.) Requiring  $v_\theta$  to be continuous at the interface:

$$v_\theta^o(R, \theta) = v_\theta^i(R, \theta)$$

Substituting (27) into (23) and (21) into (23):

$$-\left(-2\frac{c_1^o}{R^3} + \frac{1}{2}U\right)\sin\theta = -2c_4^i R^2 \sin\theta$$

or: 
$$-4c_4^i R^5 = 4c_1^o - UR^3 \quad (29)$$

Matching (25) and (28) at  $r=R$ : 
$$c_1^o = \underbrace{\frac{\mu^i}{\beta}}_{\beta} c_4^i R^5 \quad (30)$$

(30) into (29): 
$$c_4^i = \frac{U}{4(\beta+1)R^2}$$

Then (30) yields 
$$c_1^o = \frac{1}{4} \underbrace{\frac{\beta}{\beta+1}}_{\alpha} UR^3$$

With these values of the integrating constants, the streamfunctions become:

for  $r < R$ : 
$$\psi^i(r, \theta) = \frac{1}{4}(\alpha-1)R^2 U \left[ \left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4 \right] \sin^2 \theta \quad (31)$$

for  $r > R$ : 
$$\psi^o(r, \theta) = \frac{1}{2}R^2 U \left[ \frac{\alpha}{2} \left(\frac{R}{r}\right) - \left(1 + \frac{\alpha}{2}\right) \left(\frac{r}{R}\right) + \left(\frac{r}{R}\right)^2 \right] \sin^2 \theta \quad (32)$$