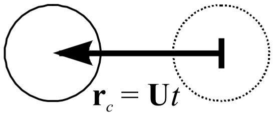
06-703 Fluid Mechanics Fall, 2001

Key to Homework #5

1a.) Consider the position of the center of the sphere. In the stationary coordinate system at some time t, the center is located at

$$\mathbf{r}_c = \mathbf{U}t$$



where $\mathbf{U}$ is the velocity of the sphere in the stationary coordinate system. In the moving coordinate system the center is always located at $\mathbf{r}_c' = \mathbf{0}$. Thus $\mathbf{r}_c' = \mathbf{r}_c - \mathbf{U}t$. Generalizing to any point (not necessarily the center of the sphere):

$$\mathbf{r}' = \mathbf{r} - \mathbf{U}t \tag{1}$$

We defined the z-axis (i.e. $\mathbf{k}$) to be pointing in the direction of motion of undisturbed fluid in the moving reference frame (to the right) whereas $\mathbf{U}$ points in the direction of motion of the sphere in the stationary reference fram (to the left). Thus

$$\mathbf{U} = -U\mathbf{k} \tag{2}$$

where $U > 0$ is the scalar magnitude of the vector $\mathbf{U}$. Substituting (2) into (1):

$$\mathbf{r}' = \mathbf{r} + Ut\,\mathbf{k}$$

Expanding this equation into its scalar components in Cartesian coordinates

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= z + Ut \end{aligned}$$

1b.) The velocities in the two reference frames are related by

$$\mathbf{v}' = \mathbf{v} - \mathbf{U} \tag{3}$$

so that the potentials are related by

$$\nabla\phi' = \nabla\phi - \mathbf{U} \tag{4}$$

Substituting (2) into (4):

$$\nabla\phi' = \nabla\phi + U\mathbf{k}$$

Expanding this equation into its scalar components in Cartesian coordinates

$$\frac{\partial\phi'}{\partial x} = \frac{\partial\phi}{\partial x}$$
$$\frac{\partial\phi'}{\partial y} = \frac{\partial\phi}{\partial y}$$
$$\frac{\partial\phi'}{\partial z} = \frac{\partial\phi}{\partial z} + U$$

Integrating we obtain $\phi' = \phi + Uz + c(t)$

In spherical coordinates $\phi' = \phi + Ur\cos\theta + c(t)$ (5)

In class we found that $\phi' = U\left(r + \frac{1}{2}\frac{R^3}{r^2}\right)\cos\theta$ (6)

Substituting (6) into (5) and solving for ϕ:

$$\phi = \frac{UR^3}{2r^2}\cos\theta - c(t)$$

1c.) Bernoulli's equation in general contains the partial time derivative: $\partial\phi/\partial t$. In the moving reference frame, this partial derivative is

$$\left(\frac{\partial\phi'}{\partial t}\right)_{\mathbf{r}'} = 0 \qquad (7)$$

because at a fixed point $\mathbf{r}'$ (relative to the moving sphere), the flow is steady: at a fixed distance from a moving sphere (i.e. $\mathbf{r}'$ fixed), the fluid velocity is not dependent on time. However, at a fixed point in space (relative to the stationary laboratory reference frame; i.e. $\mathbf{r}$ fixed), the local fluid velocity goes from zero (before the disturbance) to non-zero (during the disturbance) and back to zero (after the disturbance). In mathematical terms, this means that

$$\left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}} \neq 0$$

But what is its nonzero value? If we accept that potential flow in the stationary reference frame is unsteady we can write

$$\phi = \phi(\mathbf{r}, t)$$

and the total differential can be written as

$$d\phi = \left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}} dt + d\mathbf{r}\cdot\nabla\phi$$

Divide both sides by dt and evaluating the resulting derivatives along a path on which $\mathbf{r}'$ is held fixed:

$$\underbrace{\left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}'}}_{0} = \left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}} + \underbrace{\left(\frac{d\mathbf{r}}{dt}\right)_{\mathbf{r}'}}_{\mathbf{U}} \cdot \underbrace{\nabla\phi}_{\mathbf{v}}$$

From (1): $$\left(\frac{d\mathbf{r}}{dt}\right)_{\mathbf{r}'} = \mathbf{U}$$

Also substituting (7) and $\nabla\phi = \mathbf{v}$, we have

$$\left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}} = -\mathbf{U}\cdot\mathbf{v} \tag{8}$$

1d.) Bernoulli's equation in the stationary reference frame is

$$\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + \phi_g + \frac{p}{\rho} = \text{const.} = \phi_g + \frac{p_h}{\rho} \tag{9}$$

We evaluate the constant by requiring that far from the disturbance, the pressure tends toward hydrostatic equilibrium, the velocity vanishes and the partial derivative of potential also vanishes (from (8) when $\mathbf{v} = \mathbf{0}$):

as $r\to\infty$: $p\to p_h$, $v^2 \to 0$ and $\left(\partial\phi/\partial t\right)_{\mathbf{r}} \to 0$

Substituting these values into evaluate the integration constant (results are already shown as the right-hand side of (9)), then solving (9) for p:

$$p = p_h - \rho\frac{\partial\phi}{\partial t} - \frac{\rho}{2}v^2$$

Substituting (8): $$p = p_h + \rho\mathbf{U}\cdot\mathbf{v} - \frac{\rho}{2}\left(\mathbf{v}\cdot\mathbf{v}\right) \tag{10}$$

Bernoulli's equation in the moving reference frame in which the flow is steady and $\left(\partial\phi'/\partial t\right)_{\mathbf{r}'} = 0$ is

$$\frac{v'^2}{2} + \phi_g + \frac{p'}{\rho} = \text{const.} = \frac{U^2}{2} + \phi_g + \frac{p_h}{\rho}$$

Once again, far from the disturbance, the pressure tends toward hydrostatic equilibrium:

as $r\to\infty$: $p'\to p_h$ and $v'^2 \to U^2$

Solving for p': $$p' = p_h + \frac{\rho}{2}\left(U^2 - v'^2\right) \tag{11}$$

To compare (10) and (11) we use (3):

$$v'^2 = \mathbf{v}' \cdot \mathbf{v}' = \left(\mathbf{v} - \mathbf{U}\right) \cdot \left(\mathbf{v} - \mathbf{U}\right) = \mathbf{v} \cdot \mathbf{v} - 2\mathbf{U} \cdot \mathbf{v} + \mathbf{U} \cdot \mathbf{U} = v^2 - 2\mathbf{U} \cdot \mathbf{v} + U^2 \tag{12}$$

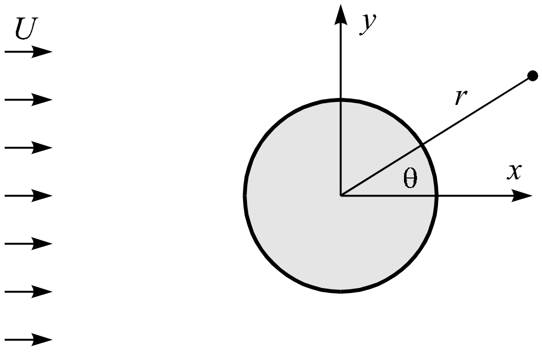
Substituting (12) into (11):

$$p' = p_h + \frac{\rho}{2}\left(2\mathbf{U} \cdot \mathbf{v} - v^2\right) = p_h + \rho \mathbf{U} \cdot \mathbf{v} - \frac{\rho}{2}\left(\mathbf{v} \cdot \mathbf{v}\right)$$

Thus (10) and (11) are identical.

2a.) For potential flow, the velocity profile is given by $\mathbf{v} = \nabla\phi$, where ϕ is chosen to satisfy Laplace's equation (see p46 of 2000 Notes):

$$\nabla^2 \phi = 0$$



In a reference frame in which the cylinder remains stationary, the flow appears to approach the cylinder at right angles to its axis (see figure at right). This is called "planar flow" because the streamlines remain in one plane (the *xy*-plane). There is no *z*-component of velocity and the *x*- and *y*-components are independent of *z*. This means that $\phi = \phi(x,y)$. While this problem could be solved in Cartesian coordinates, it is a little easier in cylindrical coordinates. In Cylindrical Coordinates, $\phi = \phi(r,\theta)$ and Laplace's equation becomes (see p739 of BS&L):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

The b.c.'s for this P.D.E. are formulated as for flow around a sphere (see p49f of Notes):

at $r = R$: $\mathbf{n} \cdot \mathbf{v} = v_r = 0$ b.c. #1

as $r \to \infty$: $\mathbf{v} \to U\mathbf{i}$ b.c. #2

What remains is to translate these b.c.'s in terms of the potential ϕ. b.c. #1 just implies that $\partial\phi/\partial r = 0$. To translate b.c. #2, we equate cooresponding components:

$$v_x = \frac{\partial \phi}{\partial x} = U$$

$$\left.\begin{aligned} v_y &= \frac{\partial \phi}{\partial y} = 0 \\ v_z &= \frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\} \Rightarrow \phi = \phi(x)$$

The 2nd and 3rd equations tell us that ϕ depends only on x. Integrating the first equation

as $r \to \infty$: $\phi \to Ux = Ur\cos\theta$ b.c. #2

To summarize, we must solve Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{13}$$

subject to $\phi \to Ux = Ur\cos\theta$ as $r \to \infty$ (14)

and $\dfrac{\partial \phi}{\partial r} = 0$ at $r = R$ (15)

Like potential flow around a sphere, guessing the trivial solution ($\phi = 0$ for all r,θ) points to (14), which suggests the guess:

$$\phi = f(r)\cos\theta \tag{16}$$

Substituting (16) into (13), we find that both terms are proportional to $\cos\theta$, which can be cancelled out, leaving:

$$r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} - f = 0 \tag{17}$$

(14) becomes: $f \to Ur$ as $r \to \infty$ (18)

(15) becomes: $\dfrac{df}{dr} = 0$ at $r = R$ (19)

(17) is a Cauchy-Euler equation, which has at least one solution of the form $f = Ar^n$. Substituting this form into (17) yields the following "characteristic" equation for n:

$$n(n-1) + n - 1 = 0$$

or

$$(n+1)(n-1) = 0$$

Thus $n = \pm 1$. We can then construct the general solution of (17) as

$$f(r) = Ar^{-1} + Br$$

To satisfy (18), we must choose $B = U$, while A is chosen to satisfy (19). The particular solution turns out to be

$$f(r) = UR\left[\left(\frac{r}{R}\right) + \left(\frac{R}{r}\right)\right]$$

(16) yields
$$\phi(r) = UR\left[\left(\frac{r}{R}\right) + \left(\frac{R}{r}\right)\right]\cos\theta$$

The velocity components are computed by taking the gradient of ϕ. The formulas for the r- and θ-components in cylindrical coordinates can be found on p739 of BS&L:

$$\boxed{\begin{aligned} v_r &= \frac{\partial\phi}{\partial r} = U\left[1 - \left(\frac{R}{r}\right)^2\right]\cos\theta \\ v_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left[1 + \left(\frac{R}{r}\right)^2\right]\sin\theta \end{aligned}} \tag{20}$$

2b.) Once the velocity profile is known, the pressure can be determined from Bernoulli's equation (neglecting gravity, ϕ_g):

$$\nabla\left(\frac{v^2}{2} + \frac{p}{\rho}\right) = \mathbf{0}$$

or
$$\frac{v^2}{2} + \frac{p}{\rho} = \text{const}$$

As $r \to \infty$, the pressure becomes p_0 while the velocity is U. Using this to evaluate the "const", the above equation becomes

$$\frac{v^2}{2} + \frac{p}{\rho} = \frac{U^2}{2} + \frac{p_0}{\rho}$$

or
$$p = p_0 + \tfrac{1}{2}\rho\left(U^2 - v^2\right) \tag{21}$$

where
$$v^2 = v_r^2 + v_\theta^2$$

To evaluate the pressure on the surface of the cylinder, we substitute (20) with $r=R$:

$$v^2\Big|_{r=R} = 0^2 + (-2U\sin\theta)^2 = 4U^2\sin^2\theta$$

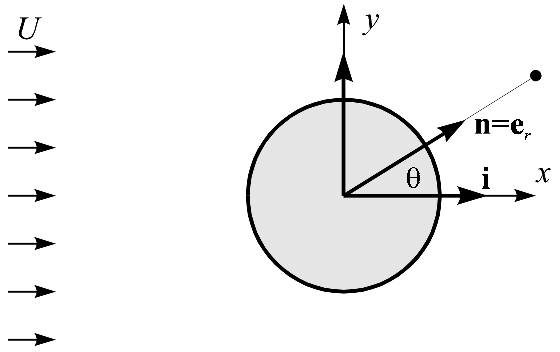
(21) becomes:
$$\boxed{p(R,\theta) = p_0 + \tfrac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)} \tag{22}$$

d'Alembert's paradox concerns the surface (drag) force due to pressure:

$$\mathbf{F}_p = -\int_A \mathbf{n} p da \tag{23}$$

Anticipating that any drag force will act in the direction of flow (i.e. in the x-direction), we dot both side of this equation by $\mathbf{i}$ to calculate this component:

$$F_{p,x} = \mathbf{i}\cdot\mathbf{F}_p = -\int_A \underbrace{\mathbf{i}\cdot\mathbf{n}}_{\cos\theta}\, p\, \underbrace{da}_{LR\,d\theta}$$



Since p is independent of z, we will select as da a strip on the surface of the cylinder, which is of length L along the z-axis and of width $R\ d\theta$ which is the arc length subtended by a small displacement $d\theta$ in θ. Since a constant pressure gives rise to no net force, we will drop the p_0 in (21), leaving

$$F_{p,x} = -\tfrac{1}{2}\rho U^2 LR \int_0^{2\pi} \left(1-4\sin^2\theta\right)\underbrace{\cos\theta\, d\theta}_{d\sin\theta} \overset{u=\sin\theta}{=} -\tfrac{1}{2}\rho U^2 LR \int_0^0 \left(1-4u^2\right)du = 0 \tag{24}$$

Already from the limits of integration, we can see this integral vanishes. Thus flow exerts no drag force on the cylinder: <u>d'Alembert's paradox results again</u>.

3.) On the lower half of the cylinder, the velocity is everywhere the same as at $r \to \infty$; thus according to Bernoulli's equation (21), the pressure is everywhere the same as at $r \to \infty$:

for $y < 0$ (or $\pi \le \theta \le 2\pi$): $p = p_0$

But a uniform pressure p_0 around the cylinder produces no force: only the deviation from p_0 produces a force and the deviations only occur on the top half of the cylinder. If the deviations from p_0 are the same as in the previous problem, we can use (24), except changing the limits of integration so as to only include the top half of the cylinder:

$$F_{p,x} = -\tfrac{1}{2}\rho U^2 LR \int_0^{\pi} \left(1-4\sin^2\theta\right)\underbrace{\cos\theta\, d\theta}_{d\sin\theta} \overset{u=\sin\theta}{=} -\tfrac{1}{2}\rho U^2 LR \int_0^0 \left(1-4u^2\right)du = 0$$

Thus <u>potential flow still predicts no drag</u> (d'Alembert's paradox), but lift would act in the y-direction. Dotting both sides of (23) by $\mathbf{j}$, we obtain $\mathbf{j}\cdot\mathbf{n} = \sin\theta$ instead of $\mathbf{i}\cdot\mathbf{n} = \cos\theta$:

$$F_{p,y} = -\tfrac{1}{2}\rho U^2 LR \int_0^{\pi} \underbrace{\left(1 - 4\sin^2\theta\right)}_{4\cos^2\theta - 3} \underbrace{\sin\theta\, d\theta}_{-d\cos\theta} \overset{u=\cos\theta}{=} \tfrac{1}{2}\rho U^2 LR \int_1^{-1} \left(4u^2 - 3\right) du$$

$$= \tfrac{1}{2}\rho U^2 LR \left[\underbrace{\left(\tfrac{4}{3}u^3\right)\Big|_1^{-1}}_{-\frac{8}{3}} \underbrace{-\left(3u\right)\Big|_1^{-1}}_{6} \right] = \tfrac{5}{3}\rho U^2 LR$$

The total force (ignoring gravity) is

$$\boxed{\mathbf{F}_p = \tfrac{5}{3}\rho U^2 LR\mathbf{j}}$$