

Key to Homework #10

1. Let $h_1 = h_2(1+\varepsilon)$. Then

$$\frac{h_1}{h_2} = 1 + \varepsilon$$

$$\ln \frac{h_1}{h_2} = \ln(1 + \varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \dots$$

$$\frac{2(h_1 - h_2)}{h_1 + h_2} = \frac{2\varepsilon}{2 + \varepsilon} = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{4}\varepsilon^3 - \dots$$

$$\ln \frac{h_1}{h_2} - \frac{2(h_1 - h_2)}{h_1 + h_2} \approx \left(\frac{1}{3} - \frac{1}{4}\right)\varepsilon^3 = \frac{1}{12}\varepsilon^3$$

$$h_1 - h_2 = h\varepsilon$$

$$\frac{1}{(h_1 - h_2)^2} \left[\ln \frac{h_1}{h_2} - \frac{2(h_1 - h_2)}{h_1 + h_2} \right] \approx \left(\frac{1}{h\varepsilon}\right)^2 \frac{1}{12}\varepsilon^3 = \frac{\varepsilon}{12h^2}$$

Substituting this result into the expression for the lift force:

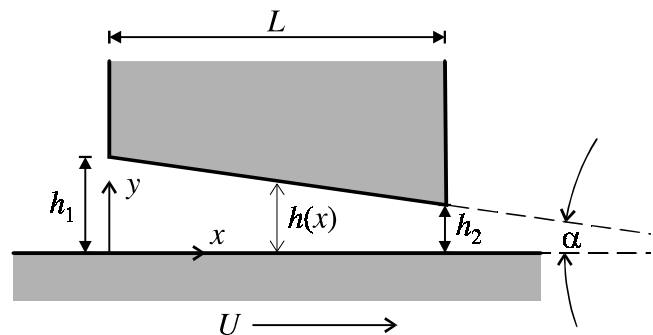
$$\frac{F_y}{W} = 6\mu UL^2 \frac{\varepsilon}{12h^2} = \frac{1}{2}\mu U \left(\frac{L}{h}\right)^2 \varepsilon$$

which is almost identical to the formula on p119 of the notes, except that — in place of ε , — the notes have α .

Recall that α is the angle by which the two surfaces are nonparallel (see the figure at right). As drawn, this angle is related to the slope of the upper surface:

$$\tan \alpha = -\frac{dh}{dx} = -\frac{h_2 - h_1}{L - 0}$$

$$\alpha \approx \frac{h_1 - h_2}{L} = \frac{h_2}{L} \varepsilon$$



so

$$\varepsilon = \frac{L}{h} \alpha$$

When α (or ε) is very small, then $\tan \alpha \approx \alpha$. This approximation was made to go from the first to the second equation above. In terms of α , the force becomes

$$\frac{F_y}{W} = \frac{1}{2} \mu U \left(\frac{L}{h} \right)^3 \alpha$$

2.) For pure squeezing flow, Reynolds lubrication equation reduces to

$$\nabla_s \cdot (h^3 \nabla_s p) = -12\mu U$$

In cylindrical coordinates, this becomes

$$\frac{1}{r} \frac{d}{dr} \left(rh^3 \frac{dp}{dr} \right) = -12\mu U$$

Multiplying through by r and integrating with respect to r :

$$rh^3 \frac{dp}{dr} = -6\mu Ur^2 + c_1$$

To keep the pressure gradient finite at $r=0$, we must choose $c_1=0$, leaving

$$dp = -6\mu U \frac{r dr}{h^3} = -6\mu UR \frac{dh}{h^3}$$

Using the hint in the second equation above, we transform $r dr = R dh$. Integrating from $r=\infty$ (where $p=p_\infty$ and $h=\infty$) to some arbitrary r :

$$\int_{p_\infty}^{p(r)} dp = -6\mu UR \int_{\infty}^{h(r)} \frac{dh}{h^3}$$

or

$$p(r) - p_\infty = \frac{3\mu UR}{h^2(r)} \quad (1)$$

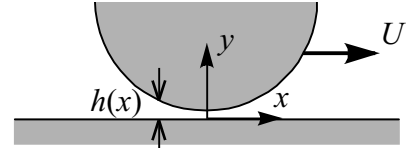
To evaluate the force, we integrate the pressure over the surface of the plate:

$$F_z = \int_0^\infty [p(r) - p_\infty] 2\pi r dr = \int_\delta^\infty \frac{3\mu UR}{h^2} 2\pi R dh = 6\pi\mu UR^2 \int_\delta^\infty \frac{dh}{h^2} = 6\pi\mu UR^2 \left(\frac{-1}{h} \right) \Big|_\delta^\infty$$

$$F_z = 6\pi\mu UR \frac{R}{\delta}$$

Once again, we have transformed an integral with respect to r into an integral with respect to h using $r dr = R dh$. See the [2000 Notes](#) on page [164](#) for an explanation of why the upper limit of integration was taken as ∞ .

- 3.) The main resistance to flow arises from the sliding motion between the sphere and the tube wall. Although the tube wall is curved (radius $R+\delta$), the radius of curvature is very large compared to the minimum thickness δ of the fluid film separating the sphere and the tube. For $\delta \ll R$, we should be able to neglect the curvature of the tube wall. Thus we flatten out the tube wall and calculate the force between a cylinder of radius R and a flat plate, with the length of the cylinder equaling the circumference of the sphere: $W = 2\pi R$. The gap thickness profile near the narrow throat is given by



$$h(x) = \delta + \frac{x^2}{2R}$$

The key to calculating the force is first finding the pressure profile from Reynolds lubrication equation. In this 2-D flow, we expect pressure will vary only with x

$$\frac{d}{dx} \left(h^3 \frac{dp}{dx} \right) = -6\mu\Delta U \frac{d\Delta h}{dx} \quad (2)$$

where

$$\Delta U = U_2 - U_1 = U - 0 = U$$

and

$$\Delta h = h_2 - h_1 = h - 0 = h(x)$$

where the subscript “2” refers to the upper body (i.e. the cylinder) and “1” refers to the lower body (i.e. the plate). RLE reduces to the following ordinary differential equation:

$$\frac{d}{dx} \left(h^3 \frac{dp}{dx} \right) = -6\mu U \frac{dh}{dx}$$

This can be formally integrated once to obtain:

$$h^3 \frac{dp}{dx} = -6\mu U h + c$$

where c is some integration constant. Dividing by h^3 :

$$\frac{dp}{dx} = -\frac{6\mu U}{h^2} + \frac{c}{h^3} \quad (3)$$

Integrating again from $x=-\infty$ (where $p=p_\infty$ and $h=\infty$) to some arbitrary x :

$$p(x) - p_\infty = -6\mu U \int_{-\infty}^x \frac{dx'}{h^2} + c \int_{-\infty}^x \frac{dx'}{h^3}$$

To evaluate the integration constant c , we require that the pressure far upstream (i.e. at $x \rightarrow +\infty$) is the same as far downstream (i.e. $p=p_\infty$):

$$0 = -6\mu U \int_{-\infty}^{\infty} \frac{dx}{h^2} + c \int_{-\infty}^{\infty} \frac{dx}{h^3}$$

or

$$c = 6\mu U \int_{-\infty}^{\infty} \frac{dx}{h^2} \bigg/ \int_{-\infty}^{\infty} \frac{dx}{h^3} = \frac{\pi\sqrt{2}}{2\delta} \frac{2\delta}{3\pi\sqrt{2}} = 6\mu U \frac{4}{3} \delta = 8\mu U \delta$$

Thus (3) becomes:

$$\frac{dp}{dx} = -\frac{6\mu U}{h^2} + \frac{8\mu U \delta}{h^3} = \frac{6\mu U}{h^2} \left(\frac{4}{3} \frac{\delta}{h} - 1 \right) \quad (4)$$

To calculate the force on the cylinder, we will need the velocity profile in the gap so we can compute the shear stress. The velocity profile in the gap is a linear combination of linear shear-flow and pressure-driven flow:

$$v_x(y) = U \frac{y}{h} + \frac{1}{2\mu} \frac{dp}{dx} y(y-h) \quad (5)$$

The force on the plate is

$$\mathbf{F} = \int_{\text{plate}} \mathbf{n} \cdot \underline{\underline{\mathbf{T}}} da$$

where $\mathbf{n} = \mathbf{e}_y$. In particular, we are interested in the x-component of the force on the plate at $y=0$:

$$F_x = \int_{\text{plate}} \mathbf{e}_y \cdot \underline{\underline{\mathbf{T}}} \cdot \mathbf{e}_x da = \int_{\text{plate}} T_{yx} da = W \int_{-\infty}^{+\infty} \tau_{yx} dx = W\mu \int_{-\infty}^{+\infty} \left. \frac{dv_x}{dy} \right|_{y=0} dx \quad (6)$$

Differentiating (5):

$$\left. \frac{dv_x}{dy} \right|_{y=0} = \frac{U}{h} - \frac{h}{2\mu} \frac{dp}{dx}$$

Substituting (4):

$$\left. \frac{dv_x}{dy} \right|_{y=0} = \frac{U}{h} - \frac{h}{2\mu} \left(-\frac{6\mu U}{h^2} + \frac{8\mu U \delta}{h^3} \right) = \frac{U}{h} + \frac{3U}{h} - \frac{4U\delta}{h^2} = \frac{4U}{h} - \frac{4U\delta}{h^2}$$

Integrating

$$\int_{-\infty}^{+\infty} \left. \frac{dv_x}{dy} \right|_{y=0} dx = 4U \int_{-\infty}^{+\infty} \frac{dx}{h} - 4U\delta \int_{-\infty}^{+\infty} \frac{dx}{h^2} = 4U\pi \sqrt{\frac{2R}{\delta}} - 4U\delta \frac{\pi}{2\delta} \sqrt{\frac{2R}{\delta}} = 2\pi U \sqrt{\frac{2R}{\delta}}$$

Substituting back into (6):

$$F_x = 2\pi\mu WU\sqrt{\frac{2R}{\delta}}$$

Finally, we substituting the perimeter of the sphere $2\pi R$ for W :

$$F_z = 4\pi^2\mu UR\sqrt{\frac{2R}{\delta}}$$

- 4a) We will use Eq. (202) on page 194 of the 2000 Notes, which represents Prandtl's Universal Law of Friction. To evaluate the friction factor f , we will need to first evaluate the Reynolds number:

$$\text{Re} \equiv \frac{2\langle \bar{v}_z \rangle R}{\nu} = 10^5$$

Armed with the value of the Reynolds number, we can find the friction factor as the root of Eq. (202). This was done by trial-and-error and the result is

$$f = 0.0045 = \frac{\tau_0}{\frac{1}{2}\rho\langle \bar{v}_z \rangle^2} = 2\left(\frac{v^*}{\langle \bar{v}_z \rangle}\right)^2$$

from which we can calculate either the friction velocity

$$v^* = \langle \bar{v}_z \rangle \sqrt{\frac{f}{2}} = 9.5 \text{ cm/s}$$

or the wall shear stress: $\tau_0 = \frac{1}{2}\rho\langle \bar{v}_z \rangle^2 f = \rho v^{*2} = 9 \text{ Pa}$

The relationship between wall shear stress and pressure drop is given in Eq. (193) of the 2000 Notes:

$$-\frac{dp}{dz} = \frac{2}{R}\tau_0 = 720 \frac{\text{Pa}}{\text{m}} = 0.032 \frac{\text{psi}}{\text{ft}}$$

- 4b.) The laminar sublayer corresponds to $0 < y^+ < 5$, where y^+ is the dimensionless distance from the wall.

$$y^+ \equiv \frac{v^*}{\nu} y$$

The thickness of the laminar sublayer would be the value of y at which $y^+ = 5$:

$$y = 5 \frac{\nu}{v^*} = 5.3 \times 10^{-5} \text{ m}$$