Homework Set #9

(due November 28, 2001)

1.) Consider the flow of fluid near an infinite flat plate (let surface be the y=0 plane) which is suddenly accelerated from rest and moves in its own plane (say, in the x-direction) with steady speed U.

a. Look for a solution of the form: $\mathbf{v} = v_x(y,t)\mathbf{e}_x$. Formulate the mathematical problem as a partial differential equation with the appropriate boundary and initial conditions.

Answer: see p142 of 2000 Notes.

b. The "similarity transform" is a mathematical technique which can change a partial differential equation into an ordinary one. This technique is often successful in transport problems which do *not* have a "natural" length scale (i.e. boundary conditions are either at 0 or ∞). When it works, the similarity transform usually has a form in which the independent variables are combined together to form a single new independent variable. Look for a solution to the problem formulated in part a) which has the form:

$$v_x = g(\eta),$$

where
$$\eta(y,t) = \frac{y}{f(t)}.$$

Carefully apply the Chain Rule to express each partial derivative in the differential equation in terms of $f(t)$ and ordinary derivatives of $f(t)$ and $g(\eta)$. Expand partial derivatives of η with respect to either y or t.

c. Eliminate any explicit appearance of y in the equation by substituting $y=\eta f$, then choose the function $f(t)$ such that η and g are the only variables appearing in the differential equation. With a suitable choice of $f(t)$, the differential equation becomes: $g'' + 2\eta g' = 0$.

d. Express the initial condition and two boundary conditions in terms of η. Choose the integration constant in the expression for $f(t)$ such that these *three* conditions map into just *two* conditions on $g(\eta)$, one at η=0 and one at η=∞. Express the solution of this boundary-value problem in terms of a definite integral.

e. The moving plate can be thought of as a source of momentum which diffuses into the initially stagnant fluid. Let δ denote the distance the momentum has diffused in time t (i.e. the "penetration depth"). Calculate $\delta(t)$ from the solution above as that value of y at which v_x equals $0.01U$.

Answer: $\delta = 3.65\sqrt{\nu t}$

2.) A similarity transform can also be used in solving Prandtl's boundary layer equations for 2-D flow problems with a stream function.

a. Transform each term of Prandtl's boundary-layer equation for a flat plate [see eq. (135) on page 140 of 2000 Notes] using

$$\psi(x,y) = \sqrt{\nu U x}\, f(\eta) \quad \text{and} \quad \eta(x,y) = \frac{y}{\sqrt{\nu x / U}}$$

where U is the inner limit of the potential flow solution. The result should be an ODE in $f(\eta)$ which does not involve any explicit dependence on x or y.

b. Now transform the boundary conditions. You should now have

$$2f''' + ff'' = 0$$

subject to $$f = f' = 0 \quad \text{at} \quad \eta = 0$$

and $$f' \to 1 \quad \text{as} \quad \eta \to \infty$$

c. The solution to this O.D.E. can be obtained numerically. The result is given in the attached Table 7.1 (taken from Schlichting, 6th ed.). Use this numerical solution to obtain the following expression for the drag coefficient:

$$C_D \equiv \frac{\dfrac{F_x}{2Wx}}{\frac{1}{2}\rho U^2} = \frac{1.328}{\sqrt{\text{Re}}}$$

where $$\text{Re} \equiv \frac{Ux}{\nu}$$

3.) Develop the boundary-layer equations for <u>uniform flow over a sphere</u> as Re→∞. Use matched asymptotic expansions in the manner outlined in the section of the 2000 Notes on pages 125-130.

Answer:

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) - \mu \frac{\partial^2 v_x}{\partial y^2} = -\frac{dp_0}{dx}$$

$$\frac{1}{\sin\frac{x}{R}} \frac{\partial}{\partial x}\left(v_x \sin\frac{x}{R}\right) + \frac{\partial v_y}{\partial y} = 0$$

where $v_x = -v_\theta$, $v_y = v_r$, $x = R(\pi - \theta)$, and $y = r - R$.

Table 7.1
from Schlichting, *Boundary Layer Theory*, 6th edition
McGraw-Hill, New York, 1968.

$\eta = y\sqrt{U/\nu x}$	f	f'	f''
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.6	0.05974	0.19894	0.33008
0.8	0.10611	0.26471	0.32739
1.0	0.16557	0.32979	0.32301
1.2	0.23795	0.39378	0.31659
1.4	0.32298	0.45627	0.30787
1.6	0.42032	0.51676	0.29667
1.8	0.52952	0.57477	0.28293
2.0	0.65003	0.62977	0.26675
2.2	0.78120	0.68132	0.24835
2.4	0.92230	0.72899	0.22809
2.6	1.07252	0.77246	0.20646
2.8	1.23099	0.81152	0.18401
3.0	1.39682	0.84605	0.16136
3.2	1.56911	0.87609	0.13913
3.4	1.74696	0.90177	0.11788
3.6	1.92954	0.92333	0.09809
3.8	2.11605	0.94112	0.08013
4.0	2.30576	0.95552	0.06424
4.2	2.49806	0.96696	0.05052
4.4	2.69238	0.97587	0.03897
4.6	2.88826	0.98269	0.02948
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
5.2	3.48189	0.99425	0.01134
5.4	3.68094	0.99616	0.00793
5.6	3.88031	0.99748	0.00543
5.8	4.07990	0.99838	0.00365
6.0	4.27964	0.99898	0.00240
6.2	4.47948	0.99937	0.00155
6.4	4.67938	0.99961	0.00098
6.6	4.87931	0.99977	0.00061
6.8	5.07928	0.99987	0.00037
7.0	5.27926	0.99992	0.00022
7.2	5.47925	0.99996	0.00013
7.4	5.67924	0.99998	0.00007
7.6	5.87924	0.99999	0.00004
7.8	6.07923	1.00000	0.00002
8.0	6.27923	1.00000	0.00001
8.2	6.47923	1.00000	0.00001
8.4	6.67923	1.00000	0.00000
8.6	6.87923	1.00000	0.00000
8.8	7.07923	1.00000	0.00000