Homework Set #7

(due November 12, 2001)

1.) A sphere of radius R is rotating about an axis drawn through the sphere's center at a steady angular velocity $\underline{\Omega}$ in a viscous fluid.

a. Find the velocity profile.

HINT: let the direction of rotation be the ϕ-direction. Let your guess for the form of the solution be guided by the form of the boundary conditions.

b. Find the torque required to rotate the sphere in a viscous fluid.

ANSWER: magnitude is $8\pi\mu\Omega R^3$

2.) A Couette-Hatschek viscometer consists of two concentric cylinders of radii R_i and R_o and length L, with the fluid (whose viscosity is to be measured) completely filling the space between them (see BSL p94). Suppose that $R_i<R_o<<L$ so that end effects are negligible.

a. Find the velocity profile if the two cylinders turn with angular velocities $\underline{\Omega}_i$ and $\underline{\Omega}_o$, which are parallel to the axes of the two cylinders.

Answer: $$v_\theta(r) = \frac{R_i^2\Omega_i - R_o^2\Omega_o}{R_i^2 - R_o^2} r + \frac{R_i^2 R_o^2(\Omega_o - \Omega_i)}{R_i^2 - R_o^2}\frac{1}{r}$$

b. Find the torques ($\underline{T}_i$ and $\underline{T}_o$) which must be applied to each of the two cylinders to rotate the inner cylinder at speed Ω_i while keeping the outer cylinder stationary (Ω_o = 0).

3.) Obtain the <u>general solution</u> to $E^2(E^2\psi)=0$ for creeping flow around a sphere of radius R in terms of the stream function $\psi(r,\theta) = f(r)\sin^2\theta$.

Answer: $$\psi(r,\theta) = \left(c_1 r^{-1} + c_2 r + c_3 r^2 + c_4 r^4\right)\sin^2\theta$$

4.) Consider the problem of uniform flow (with $\mathbf{v}\to\mathbf{U} = U\mathbf{e}_x$ as $r\to\infty$) normal to the axis of a circular cylinder in the limit of $Re\to 0$. Let the axis of the cylinder coincide with the z-axis. Then we have 2-D flow (v_z=0 and $\partial/\partial z$=0).

a. Proceeding as we did in class for creeping flow around a sphere (start with curl$^3\mathbf{v}$ = $\mathbf{0}$), show that the streamfunction $\psi(r,\theta)$ must satisfy

$$\nabla^2(\nabla^2\psi) = 0$$

Also show that the appropriate boundary conditions are as $r\to\infty$: $\psi \to Ur\sin\theta$

at $r=R$: $\dfrac{\partial\psi}{\partial\theta} = \dfrac{\partial\psi}{\partial r} = 0$

b. *Try* to find a solution of the form

$$\psi(r,\theta) = f(r)\sin\theta$$

In particular, show that this form satisfies the differential equation but cannot simultaneously satisfy both boundary conditions at $r\to\infty$ and at $r=R$. This is known as "Stokes Paradox."

Hint #1: The solution of the associated 4th order Cauchy-Euler problem has a repeated root in this case. Thus only three solutions are of the form r^n (n turns out to be −1, +1 or +3); the 4th linearly independent solution of the O.D.E. has the form $r\ln r$ as determined from "variation of parameters" (see p445 of Greenberg).

Hint #2: $r\ln r$ behaves like $r^{1+\varepsilon}$ as $r\to\infty$, where ε is an arbitrarily small positive number. In other words, $r\ln r$ tends to "blow up" faster than r as $r\to\infty$.

5.) Evaluate the integration constants from Prob. 3) so as to satisfy each of the following set of boundary conditions:

a. Uniform flow over a rigid sphere:

$r\to\infty$: $\psi \to (1/2)Ur^2\sin^2\theta$

$r=R$: $v_r = v_\theta = 0$

Answer: see 2000 Notes p104, eqn 76.

b. Exterior problem ($\mu=\mu_o$ for $r\geq R$) for uniform flow over a spherical fluid drop (where v_θ at $r=R$ is unknown):

$r\to\infty$: $\psi^o \to (1/2)Ur^2\sin^2\theta$

$r=R$: $v_r^o = 0$

where the superscript "o" is used to denote the solution to the outer problem. This is same as part a) except that we do not require $v_\theta = 0$ at $r=R$ since a fluid drop is not rigid. You will have one integration constant remaining.

Answer: $$f^o(r) = \frac{c_1^o}{r} + \left(-\frac{1}{2}UR - \frac{c_1^o}{R^2}\right)r + \frac{1}{2}Ur^2$$

c. Interior problem ($\mu=\mu_i$ for $r\leq R$) for uniform flow over a spherical fluid drop:

$r=R$: $v_r^i=0$

$r=0$: v_r^i, v_θ^i are bounded where the superscript "*i*" is used to denote the solution to the inner problem. Again you will have one integration constant remaining.

Answer: $$f^i(r) = c_4^i\left(-R^2r^2 + r^4\right)$$

d. Complete problem ($0 \le r < \infty$ with $\mu_i \neq \mu_o$) for uniform flow over a spherical fluid drop. Evaluate the two remaining integration constants (from parts b and c) by matching:

$$v_\theta{}^i = v_\theta{}^o \text{ at } r=R$$

and $$\tau_{r\theta}{}^i = \tau_{r\theta}{}^o \text{ at } r=R$$

This was first accomplished by Rybczynski in 1911 (see L&L, p69-70).

Answer: $$\psi^i(r,\theta) = \frac{1}{4}(\alpha - 1)R^2U\left[\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4\right]\sin^2\theta$$

$$\psi^o(r,\theta) = \frac{1}{2}R^2U\left[\frac{\alpha}{2}\left(\frac{R}{r}\right) - \left(1 + \frac{\alpha}{2}\right)\left(\frac{r}{R}\right) + \left(\frac{r}{R}\right)^2\right]\sin^2\theta$$

where $\alpha \equiv \beta/(\beta+1)$ and $\beta \equiv \mu_i/\mu_o$.

Hint: intermediate results can be checked by comparing with part a): note that as $\beta \to \infty$ (i.e. as $\alpha \to 1$), we should get the same results a for a rigid sphere (part a).