Homework Set #6

(due October 10, 2001)

1.) Consider the problem of potential flow around a sphere of radius R. For axisymmetric flow around a sphere, the stream function is related to velocity by

$$\mathbf{v} = \text{curl}\left\{\frac{\psi(r,\theta)\mathbf{e}_\phi}{r\sin\theta}\right\}$$

a. Write down the differential equation and boundary conditions which the stream function must satisfy in this problem. In particular, show that

$$\psi \to \tfrac{1}{2}Ur^2\sin^2\theta \quad \text{as} \quad r\to\infty$$

corresponds to $\mathbf{v} \to U\mathbf{k}$ and no flow into the sphere requires $\partial\psi/\partial\theta = 0$ at $r=R$.

b. Deduce a physical meaning for the stream function in this geometry.

HINTS: what is the flowrate in a tube formed by rotating a streamline (ψ = const) about the z-axis? To answer this question, consider the boundary condition for ψ as $r\to\infty$ in which limit the tube becomes a circular cylinder.

ANSWER: $$\psi = \frac{Q}{2\pi}$$

c. Solve for the stream function $\psi(r,\theta)$ and deduce the velocity profile.

ANSWER: $$\psi(r,\theta) = \frac{1}{2}UR^2\left[\left(\frac{r}{R}\right)^2 - \left(\frac{R}{r}\right)\right]\sin^2\theta$$

2.) In the most general statement of Hooke's Law of Elastic Solids, stress and strain are related by:

$$\underline{\underline{\mathbf{T}}} = 2\eta\underline{\underline{\boldsymbol{\varepsilon}}} + \lambda(\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{I}}} \tag{1}$$

where η and λ are two independent material constants.

a. Uniaxial stress applied along the x-axis produces normal strain along all three axes, which are related according to $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$, where ν is a material property called "Poisson's ratio." Expand (1) to obtain expressions for T_{xx} and T_{yy} in terms of ε_{xx} and the material properties.

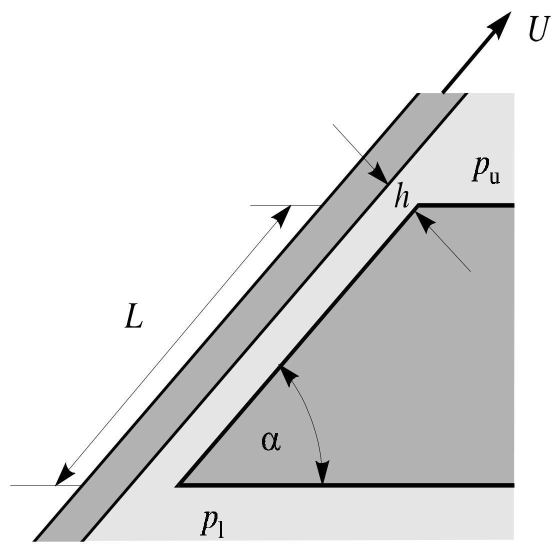
b. Recognizing that $T_{yy} = 0$ for uniaxial stress along the x-axis, find a relationship between Poisson's ratio and the material constants η and λ.

ANSWER: $$\nu = \frac{\lambda}{2(\lambda + \eta)}$$

c. Relate Young's modulus, E, for pure uniaxial stress to the material constants η and λ.

d. For the second case of pure shear, expand (1) to obtain expressions for T_{xy} in terms of ε_{xy} and the material properties. Relate G, the modulus of elasticity for pure strain, to η and λ.

e. Show that $\nu = \frac{E}{2G} - 1$.



3.) Fluid is being pumped from a lower reservoir at pressure p_l to an upper reservoir at pressure p_u by pulling a plate at speed U parallel to a second stationary plate which is inclined at angle α from the horizon (see figure at right). You may assume that $h \ll L$ so that entrance and exit effects are negligible:

$$\mathbf{v} = v_x(y)\mathbf{i}$$

but do *not* neglect gravity. Determine the following:

a. the velocity profile,

ANSWER:

$$v_x(y) = \frac{1}{2\mu}\left(\frac{P_l - P_u}{L}\right)(h-y)\,y + U\frac{y}{h}$$

where the *x*-axis is parallel to the two plates, *y* is the distance from the stationary plate, and the *P*'s are dynamic pressures.

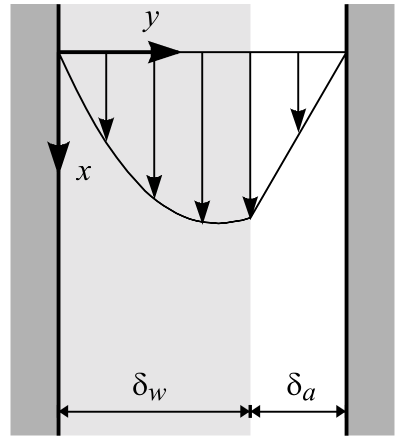
b. the volumetric flowrate per width *W* of plate,

c. the force exerted by the fluid on the moving plate,

d. the force exerted by the fluid on the stationary plate, and

e. the power required to drive the pump.

4.) In our analysis of the steady flow of a viscous liquid film down an inclined plane (see p87 of 2000 Notes), we treated the air outside the film as inviscid. This problem is designed to see when this is a good approximation.



Consider a vertical film of water in contact with a vertical film of air which, in turn is in contact with a second vertical wall, as shown in the sketch at right. In the analysis below, you may assume the isotropic pressure remains 1 atm everywhere and the density of air is virtually zero.

a. Formulate separate differential equations and boundary conditions for the velocity profile in each of the two phases. Assume that v_x at $y=\delta_w$ is known and equal to U (its value will be determined later). Find the velocity profiles in both phases.

b. Use the boundary condition on the stress at the free surface ($y=\delta_w$) to deduce the value of U.

c. Determine the flowrate of liquid in the film in terms of the film thicknesses and fluid viscosities, μ_w and μ_a.

d. How thin does the air film have to be (as a fraction of δ_w) to reduce the water flowrate by 1% (compared to an air film having $\delta_a = \infty$), taking $\mu_w = 1000\ \mu_a$?