Homework Set #4

(due September 26, 2001)

1.) Find the force due to hydrostatic pressure by explicitly evaluating the surface integral

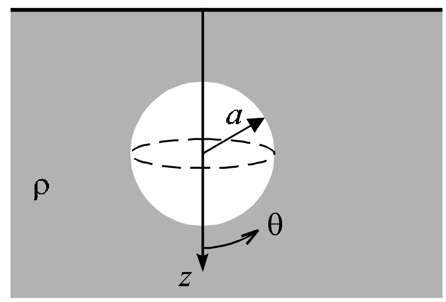
$$\mathbf{F}_p = -\int_A \mathbf{n} p \, da$$

when a sphere of radius a is immersed in a fluid of density ρ

a. having uniform pressure p_o,

b. having a pressure profile p which increases linearly with depth, according to

$$p(z) = p_o + \rho g z$$



Hints: In spherical coordinates (r,θ,ϕ) the outward pointing normal is

$$\mathbf{n} = \mathbf{e}_r(\theta,\phi) = (\sin\theta\cos\phi)\mathbf{i} + (\sin\theta\sin\phi)\mathbf{j} + (\cos\theta)\mathbf{k}$$

where θ is measured from the z-axis (see figure) and ϕ is measured around the equator The surface element is a nearly rectangular patch, subtended by increments $d\theta$ and $d\phi$ in the angles, having area

$$da = (r\sin\theta \, d\phi)(r \, d\theta)$$

Answer: $\mathbf{F}_p$ is $\mathbf{0}$ for part a) and $-\frac{4}{3}\pi a^3 \rho g \mathbf{k}$ for part b).

2.) Repeat problem 1.) using the Divergence Theorem to express the surface integrals in terms of volume integrals.

Hint: To evaluate the volume integrals, you will need to assign a pressure profile to the region inside the sphere. When the sphere is constructed of a rigid solid material, the actual pressure profile inside the solid is indeterminant. However, the surface force depends only on the pressure profile in the fluid near the surface of the sphere, which is the same if the rigid solid sphere is replaced by a fluid sphere of the same size.

3.) An incompressible fluid in a partially filled container, exposed to a gravitational field $\mathbf{g} = g\mathbf{k}$, undergoes rigid-body rotation about the z-axis with a steady angular velocity $\underline{\mathbf{\Omega}} = \Omega\mathbf{k}$.

a. Show that the corresponding translational velocity profile can be written as

$$\mathbf{v}(\mathbf{r}_p) = \underline{\mathbf{\Omega}} \times \mathbf{r}_p = \Omega r \mathbf{e}_\theta$$

where r,θ,z are cylindrical coordinates defined with an origin on the axis of rotation and $\mathbf{r}_p$ is the position vector ($\mathbf{r}_p = r\mathbf{e}_r + z\mathbf{e}_z$).

Comment: in class, we showed that $\mathbf{v}(\mathbf{r}_p) = \Omega r\mathbf{e}_\theta$. Here I want you to show that $\underline{\mathbf{\Omega}}\times\mathbf{r}_p = \Omega r\mathbf{e}_\theta$.

b. Evaluate $D\mathbf{v}/Dt$ in cylindrical coordinates.

Answer: $\frac{D\mathbf{v}}{Dt} = -r\Omega^2\mathbf{e}_r$

c. Evaluate $p(\mathbf{r}_p)$

Answer: $p(r,z) = p_0 + \frac{1}{2}\rho r^2\Omega^2 - \rho g z$

d. Recognizing that the pressure must equal p_0 (atmospheric) along the surface of the liquid, find an equation to describe the shape of the surface. Make a sketch of your solution.

e. Evaluate $\nabla\times\mathbf{v}$; express your result in terms of the *vector* $\underline{\mathbf{\Omega}}$.