Key to Final Exam

1.) Choice iv) is the only one have a boundary layer in its solution. Boundary layers are caused by a reduction in order of the differential equation upon setting the small parameter ε identically equal to zero. Choices iii) and iv) both have the highest-order derivative multiplied by a small parameter, which results in a reduction in order when the small parameter is set equal to zero. However, choice iii) is satisfied by the trivial solution $y(x,\varepsilon) = 0$ for any x or any ε.

2a.) Creeping flow is usually a good approximation when U is small enough so that

$$\boxed{\mathrm{Re} \equiv \frac{\rho U R}{\mu} < 1}$$

2b.) In the creeping flow approximat, we neglect inertial terms in the Navier Stokes equation:

$$\underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\text{inertia}} = -\nabla p + \underbrace{\mu \nabla^2 \mathbf{v}}_{\text{viscous}}$$

2c.) Dropping the inertial terms, the Navier Stokes equation become

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{v} \tag{1}$$

In addition, we need the continuity equation which, for an incompressible fluid, is

$$\nabla \bullet \mathbf{v} = 0 \tag{2}$$

2d.) The continuity equation (2).

2e.) p can be eliminated from (1) by <u>taking the curl of both sides</u>:

$$\mathbf{0} = -\nabla \times \nabla p + \mu \nabla \times \nabla^2 \mathbf{v} \tag{3}$$

Since $\nabla \times \nabla p = \mathbf{0}$ no matter what p is, we eliminate p from the equation.

2f.) In spherical coordinates, (3) will be satisfied if $\psi(r,\theta)$ satisfies:

$$\boxed{E^2\left(E^2\psi\right) = 0}$$

where

$$E^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$$

2g.) In a reference frame in which the inviscid bubble remains stationary, no-slip requires

at $r=R$: $$v_r = 0 \tag{4}$$

No-slip also requires matching v_θ across the interface, but since we don't know the velocity profile inside the bubble, this is of little help. Instead, we match the viscous shear stress across the interface. Since the viscosity inside the bubble vanishes, so must the viscous shear:

at $r=R$: $$\tau_{r\theta} = 0 \tag{5}$$

The shear stress can be related to velocity using Newton's Law of Viscosity:

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

According to (4), $v_r = 0$ everywhere on the surface, so the second term above drops, leaving us to require only

at $r=R$: $$\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) = 0 \tag{6}$$

In terms of the stream function, the velocity components are given by

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

so (4) require $$\boxed{\frac{\partial \psi}{\partial \theta} = 0} \text{ at } r=R$$

while (6) requires $$\boxed{\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi}{\partial r}\right) = 0} \text{ at } r=R$$

2h.) Once **v** is known, we can determine p by integrating (1):

$$\boxed{\nabla p = \mu \nabla^2 \mathbf{v}}$$

2i.) Yes. No-slip on a rigid stationary sphere requires all components of velocity to vanish. In the velocity profile given, $v_r = v_\phi = 0$, but $v_\theta \neq 0$. The more general statement of "no-slip" requires the velocity vector field **v** to be continuous across the interface. Since we don't know what the velocity profile of the inviscid fluid inside is, $v_\theta \neq 0$ doesn't necessarily violate no-slip. Indeed a closer look at Hwk #7, Prob. 2 in the case in which $\mu_i = 0$ would reveal that no-slip is satisfied.

2j.) Anticipating that the net force will act in the *z*-direction, we dot our general equation for force by **k**:

$$F_z = \int_A \mathbf{k}\cdot(\mathbf{e}_r\cdot\underline{\underline{\mathbf{T}}})da \tag{7}$$

Substituting
$$\underline{\underline{\mathbf{T}}} = -p\underline{\underline{\mathbf{I}}} + \underline{\underline{\tau}}$$

$$\mathbf{e}_r\cdot\underline{\underline{\mathbf{T}}} = -p(\mathbf{e}_r\cdot\underline{\underline{\mathbf{I}}}) + \mathbf{e}_r\cdot\underline{\underline{\tau}}$$

$$= -p\mathbf{e}_r + \mathbf{e}_r\cdot\underline{\underline{\tau}}$$

$$\mathbf{e}_r\cdot\underline{\underline{\tau}} = \tau_{rr}\mathbf{e}_r + \tau_{r\theta}\mathbf{e}_\theta + \tau_{r\phi}\mathbf{e}_\phi$$

For the determined velocity profile, Newton's law of viscosity in spherical coordinates yields $\tau_{r\phi}$=0 for all *r* and

at r=R:
$$\tau_{rr} = 2\mu\left.\frac{\partial v_r}{\partial r}\right|_{r=R} = \frac{2\mu U}{R}\cos\theta \tag{8}$$

at r=R:
$$\mathbf{e}_r\cdot\underline{\underline{\mathbf{T}}} = (-p + \tau_{rr})\,\mathbf{e}_r + \tau_{r\theta}\mathbf{e}_\theta$$

$$\mathbf{k}\cdot(\mathbf{e}_r\cdot\underline{\underline{\mathbf{T}}}) = (-p + \tau_{rr})(\mathbf{k}\cdot\mathbf{e}_r) + \tau_{r\theta}(\mathbf{k}\cdot\mathbf{e}_\theta)$$

Using $\mathbf{k} = \mathbf{e}_z$ from what's given:

$$\mathbf{k}\cdot(\mathbf{e}_r\cdot\underline{\underline{\mathbf{T}}}) = (-p + \tau_{rr})\cos\theta - \tau_{r\theta}\sin\theta$$

Finally, we can integrate using an azimuthal strip of width $Rd\theta$ and of radius $R\sin\theta$:

$$da = 2\pi(R\sin\theta)(Rd\theta)$$

(7) becomes:

$$F_z = 2\pi R^2\int_0^\pi\left\{\left[-p(R,\theta)+\tau_{rr}\right]\sin\theta\cos\theta - \tau_{r\theta}(R,\theta)\sin^2\theta\right\}d\theta$$

Owing to our zero-stress boundary condition (5), the second term in the integrand is zero. To (8), we add the pressure at $r=R$: $$p(R,\theta) = p_\infty - \frac{\mu U}{R}\cos\theta$$

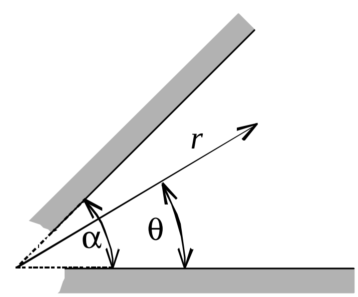
We know that integrating the constant p_∞ around the sphere leads to no force, so we drop this term, leaving

$$F_z = 2\pi R^2 \int_0^\pi \frac{3\mu U}{R}\cos^2\theta \underbrace{\sin\theta d\theta}_{-d\cos\theta} = -6\pi\mu UR \int_1^{-1} u^2\, du = -6\pi\mu UR \underbrace{\left(\frac{1}{3}u^3\right)\Big|_1^{-1}}_{-2/3}$$

$$\boxed{F_z = 4\pi\mu UR}$$

3a.) A boundary layer is a region near a boundary in which the gradient of the solution is much larger than outside the layer. In the left profile, the gradient is about of the same throughout the domain, but in the right profile the gradient near the walls is much larger than near the centerline. Therefore, the boundary layer appears in the right profile.

3b.) We choose cylindrical coordinates, with r measured from the intersection of the two planes. We do not require potential flow to satisfy no-slip, but we do require no flow through the walls, so



at $\theta=0$: $v_\theta = 0$

at $\theta=\alpha$: $v_\theta = 0$

Clearly we need an r-component of flow to supply the sink at r=0 and that r-component must depend on r to satisfy continuity. But we can satisfy the b.c.'s above if we choose $v_\theta = 0$ for all θ. So our guess is

$$v_r = v_r(r), \quad v_\theta = v_z = 0$$

In terms of a velocity potential ϕ, this means

$$\phi = \phi(r)$$

Potential flow satisfies $$\nabla^2\phi = 0$$

or in cylindrical coordinates: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

Multiplying through by r and integrating once:

$$r\frac{d\phi}{dr} = c_1 \qquad \text{or} \qquad \frac{d\phi}{dr} = \frac{c}{r}$$

Integrating again $\phi(r) = c\ln r$

or $$v_r = \frac{d\phi}{dr} = \frac{c}{r} \quad \text{and} \quad v_\theta = v_z = 0 \tag{9}$$

where the integration constant c could be related to the strength of the sink.

3c.) In potential flow, the pressure is computed from Bernoulli's equation:

$$\frac{p}{\rho} + \frac{1}{2}v^2 = \text{const} = \frac{p_\infty}{\rho} \tag{10}$$

where the constant can be evaluated from the conditions far from the sink (i.e. $r\to\infty$) where $p \to p_\infty$ and $v \to 0$. Substituting (9) into (10):

$$p = p_\infty - \frac{\rho c^2}{2r^2}$$

3d.) The velocity profile inside the boundary layer for 2-D flow is described by Prandtl's BL equations:

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} - \nu\frac{\partial^2 v_x}{\partial y^2} = U_0\frac{dU_0}{dx}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

where x is measured along the lower plate and y is measured normal to the lower plate. $U_0(x)$ is the potential flow solution:

$$U_0(x) = \frac{c}{x} \quad \text{and} \quad \frac{dU_0}{dx} = -\frac{c}{x^2}$$

Prandtl's equations become

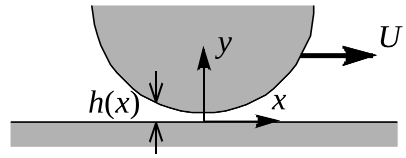
$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} - \nu\frac{\partial^2 v_x}{\partial y^2} = -\frac{c^2}{x^3}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Appropriate boundary conditions are

at y=0: $v_x = v_y = 0$

as $y \to \infty$: $$v_x \to \frac{c}{x}$$



4.) The main resistance to flow arises from the sliding motion between the sphere and the tube wall. Although the tube wall is curved (radius $R+\delta$), the radius of curvature is very large compared to the minimum thickness δ of the fluid film separating the sphere and the tube. For $\delta \ll R$, we should be able to neglect the curvature of the tube wall. Thus we flatten out the tube wall and calculate the force between a cylinder of radius R and a flat plate, with the length of the cylinder equaling the circumference of the sphere: $W = 2\pi R$. The gap thickness profile near the narrow throat is given by

$$h(x) = \delta + \frac{x^2}{2R}$$

The key to calculating the force is first finding the pressure profile from Reynolds lubrication equation. In this 2-D flow, we expect pressure will vary only with x

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) = -6\mu\Delta U \frac{d\Delta h}{dx} \tag{11}$$

where $$\Delta U = U_2 - U_1 = U - 0 = U$$

and $$\Delta h = h_2 - h_1 = h - 0 = h(x)$$

where the subscript "2" refers to the upper body (i.e. the cylinder) and "1" refers to the lower body (i.e. the plate). RLE reduces to the following ordinary differential equation:

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) = -6\mu U \frac{dh}{dx}$$

This can be formally integrated once to obtain:

$$h^3 \frac{dp}{dx} = -6\mu U h + c$$

where c is some integration constant. Dividing by h^3:

$$\frac{dp}{dx} = -\frac{6\mu U}{h^2} + \frac{c}{h^3} \tag{12}$$

Integrating again from $x=-\infty$ (where $p=p_\infty$ and $h=\infty$) to some arbitrary x:

$$p(x) - p_\infty = -6\mu U \int_{-\infty}^{x} \frac{dx'}{h^2} + c \int_{-\infty}^{x} \frac{dx'}{h^3}$$

To evaluate the integration constant c, we require that the pressure far upstream (i.e. at $x \to +\infty$) is the same as far downstream (i.e. $p = p_\infty$):

$$0 = -6\mu U \int_{-\infty}^{\infty} \frac{dx}{h^2} + c \int_{-\infty}^{\infty} \frac{dx}{h^3}$$

or

$$c = 6\mu U \int_{-\infty}^{\infty} \frac{dx}{h^2} \bigg/ \int_{-\infty}^{\infty} \frac{dx}{h^3} = \frac{\dfrac{\pi\sqrt{2}}{2\delta}}{\dfrac{3\pi\sqrt{2}}{8\delta^2}} = 6\mu U \frac{4}{3}\delta = 8\mu U \delta$$

Thus (12) becomes:

$$\frac{dp}{dx} = -\frac{6\mu U}{h^2} + \frac{8\mu U \delta}{h^3} = \frac{6\mu U}{h^2}\left(\frac{4}{3}\frac{\delta}{h} - 1\right) \qquad (13)$$

To calculate the force on the cylinder, we will need the velocity profile in the gap so we can compute the shear stress. The velocity profile in the gap is a linear combination of linear shear-flow and pressure-driven flow:

$$v_x(y) = U\frac{y}{h} + \frac{1}{2\mu}\frac{dp}{dx} y (y - h) \qquad (14)$$

The force on the plate is

$$\mathbf{F} = \int_{\text{plate}} \mathbf{n} \bullet \underline{\underline{\mathbf{T}}} \, da$$

where $\mathbf{n} = \mathbf{e}_y$. In particular, we are interested in the x-component of the force on the plate at y=0:

$$F_x = \int_{\text{plate}} \mathbf{e}_y \bullet \underline{\underline{\mathbf{T}}} \bullet \mathbf{e}_x \, da = \int_{\text{plate}} T_{yx} \, da = W \int_{-\infty}^{+\infty} \tau_{yx} \, dx = W\mu \int_{-\infty}^{+\infty} \left.\frac{dv_x}{dy}\right|_{y=0} dx \qquad (15)$$

Differentiating (14):

$$\left.\frac{dv_x}{dy}\right|_{y=0} = \frac{U}{h} - \frac{h}{2\mu}\frac{dp}{dx}$$

Substituting (13):

$$\left.\frac{dv_x}{dy}\right|_{y=0} = \frac{U}{h} - \frac{h}{2\mu}\left(-\frac{6\mu U}{h^2} + \frac{8\mu U \delta}{h^3}\right) = \frac{U}{h} + \frac{3U}{h} - \frac{4U\delta}{h^2} = \frac{4U}{h} - \frac{4U\delta}{h^2}$$

Integrating

$$\int_{-\infty}^{+\infty} \left.\frac{dv_x}{dy}\right|_{y=0} dx = 4U\int_{-\infty}^{+\infty}\frac{dx}{h} - 4U\delta\int_{-\infty}^{+\infty}\frac{dx}{h^2} = 4U\pi\sqrt{2} - 4U\delta\frac{\pi\sqrt{2}}{2\delta} = 2\pi\sqrt{2}U$$

Substituting back into (15): $F_x = 2\pi\sqrt{2}\mu WU$

Finally, we substituting the perimeter of the sphere $2\pi R$ for W:

$$\boxed{F_z = 4\pi^2\sqrt{2}\mu UR}$$