Key to Final Exam

1.) **Re << 1**

2.) An ideal fluid deforms isentropically, which means that viscous dissipation is negligible. For flow around submerged objects, this occurs when **Re >> 1**. A fluid can also be treated as ideal when it is not being deformed (e.g. solid-body rotation of a fluid).

3.) For steady flow, the criterion is that the largest velocity is small compared to the speed of sound in the fluid: $\boxed{|v_{max}| << c}$

4.) We derived Bernoulli's equation for potential flow of an ideal incompressible fluid. For flow around submerged objects, this occurs when **Re >> 1** and $\boxed{|v_{max}| << c}$.

5.) The sphere itself is undergoing rigid-body rotation at an angular velocity $\underline{\Omega} = \Omega \mathbf{e}_z$:

$$\mathbf{v} = \underline{\Omega} \times \mathbf{r}$$

where the position vector in spherical coordinates is $\mathbf{r} = r\mathbf{e}_r$. Expressing $\mathbf{e}_z$ in terms of the unit vectors in spherical coordinates:

$$\mathbf{e}_z = (\cos\theta)\mathbf{e}_r + (\sin\theta)(-\mathbf{e}_\theta)$$

Crossing the two vectors gives: $\mathbf{v} = r\Omega(\sin\theta)\mathbf{e}_\phi$

At the outer surface of the rotating sphere, no slip requires that the fluid move with the same speed as the solid sphere:

at $r=R$: $$v_\phi = R\Omega(\sin\theta) \tag{1}$$

whereas no slip on the surface of the socket requires:

at $r=R+\delta$: $$v_\phi = 0$$

For linear shear flow in the gap we estimate:

$$\frac{dv_\phi}{dr} \approx \frac{\Delta v_\phi}{\Delta r} = \frac{0 - R\Omega\sin\theta}{(R+\delta) - R} = -\frac{R\Omega\sin\theta}{\delta} \tag{2}$$

The torque on any closed surface A is

$$\mathcal{T} = \oint_A \mathbf{r} \times \left(\mathbf{n} \cdot \underline{\underline{\mathbf{T}}}\right) da \tag{3}$$

The unit normal to the sphere is $\mathbf{e}_r$:

$$\mathbf{n}\cdot\underline{\underline{\mathbf{T}}}=(\mathbf{e}_r)\cdot\left(\sum_i\sum_j T_{ij}\mathbf{e}_i\mathbf{e}_j\right)=T_{rr}\mathbf{e}_r+T_{r\theta}\mathbf{e}_\theta+T_{r\phi}\mathbf{e}_\phi$$

The lever arm drawn from the center to any point on the surface of the sphere is $\mathbf{r} = R\mathbf{e}_r$, so

$$\mathbf{r}\times\left(\mathbf{n}\cdot\underline{\underline{\mathbf{T}}}\right)=(R\mathbf{e}_r)\times\left(T_{rr}\mathbf{e}_r+T_{r\theta}\mathbf{e}_\theta+T_{r\phi}\mathbf{e}_\phi\right)=-RT_{r\phi}\mathbf{e}_\theta+RT_{r\theta}\mathbf{e}_\phi$$

We anticipate that the net torque vector will be parallel to angular velocity vector, which in turn is parallel to $\mathbf{e}_z$. So we will focus our attention on computing the z-component of the torque:

$$\mathbf{e}_z\cdot\left[\mathbf{r}\times\left(\mathbf{n}\cdot\underline{\underline{\mathbf{T}}}\right)\right]=\left[(\cos\theta)\mathbf{e}_r-(\sin\theta)\mathbf{e}_\theta\right]\cdot\left(-RT_{r\phi}\mathbf{e}_\theta+RT_{r\theta}\mathbf{e}_\phi\right)=R(\sin\theta)T_{r\phi}$$

The z-component of (3) becomes

$$\mathbf{e}_z\cdot\underline{\mathcal{T}}=R\oint_{\mathcal{A}}(\sin\theta)T_{r\phi}\,da \tag{4}$$

The stress is related to the velocity profile by:

$$T_{r\phi}=\tau_{r\phi}=\mu r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)=\mu\frac{\partial v_\phi}{\partial r}-\mu\frac{v_\phi}{r}$$

Substituting (1) and (2) for r=R:

$$T_{r\phi}=-\mu\frac{R\Omega\sin\theta}{\delta}-\mu\frac{R\Omega\sin\theta}{R}=-\mu\left(\frac{R}{\delta}+1\right)\Omega\sin\theta\approx-\mu\Omega\frac{R}{\delta}\sin\theta$$

Since the integrand depends only on θ, we choose da to be $2\pi(R\sin\theta)(Rd\theta)$. (4) becomes

$$\mathcal{T}_z=R\int_0^\pi(\sin\theta)\left(-\mu\Omega\frac{R}{\delta}\sin\theta\right)(2\pi R\sin\theta)(Rd\theta)=-2\pi\mu\Omega\frac{R^4}{\delta}\underbrace{\int_0^\pi\sin^3\theta d\theta}_{4/3}$$

$$\boxed{\mathcal{T}_z=\frac{8}{3}\pi\mu\Omega\frac{R^4}{\delta}}$$

6a.) **False**

6b.) **False**

6c.) **False**

6d.) **True**

7.) In this problem the equation of the lower surface (the plate) is just

$$h_1(r) = 0$$

The equation of the upper surface is $h_2(r) = \delta + mr$

The total gap between the two surfaces is

$$h = h_1 + h_2 = \delta + mr \tag{5}$$

For Reynolds equation, we also need

$$\Delta h = h_2 - h_1 = \delta + mr$$

The velocity of the upper surface is

$$U_2 = 0 \quad V_2 = 0 \quad W_2 = -U$$

while the lower surface is stationary:

$$U_1 = 0 \quad V_1 = 0 \quad W_1 = 0$$

The following quantities appear in Reynolds equation becomes

$$\Delta U = 0 \quad \Delta V = 0 \quad \Delta W = -U$$

Reynolds equation becomes $$\nabla_s \cdot \left(h^3 \nabla_s p \right) = -12\mu U \tag{6}$$

Because the upper surface is a surface of revolution, we expect squeezing flow to be axisymmetric in cylindrical coordinates. In other words, we expect that $p = p(r)$ (i.e. no θ-dependence). In cylindrical (r,θ,z) or polar coordinates (r,θ), (6) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r h^3 \frac{dp}{dr} \right) = -12\mu U$$

Multiplying through by r and integrating:

$$rh^3 \frac{dp}{dr} = -6\mu U r^2 + c$$

Dividing by rh^3:

$$\frac{dp}{dr} = -6\mu U \frac{r}{h^3} + \frac{c}{rh^3}$$

When this is integrated a second time, the second term will lead to a logarithmic singularity at r=0. To keep the force from becoming infinite for all δ>0, we need to choose c=0. To integrate this, we note that (5) requires $dh = m\,dr$ and $r = (h\text{-}\delta)/m$.

$$p(r) - p_\infty = -6\mu U \int_R^r \frac{r}{h^3} dr = -6\mu U \int_{h_R}^{h} \frac{(h-\delta)/m}{h^3} \frac{dh}{m} = \frac{6\mu U}{m^2} \int_h^{h_R} \frac{h-\delta}{h^3} dh$$

$$= \frac{6\mu U}{m^2} \int_h^{h_R} \left(h^{-2} - \delta h^{-3} \right) dh = \frac{6\mu U}{m^2} \left(-h^{-1} + \frac{1}{2}\delta h^{-2} \right)\Bigg|_{h(r)}^{h_R}$$

$$= \frac{6\mu U}{m^2} \left(\frac{\delta - 2h}{2h^2} \right)\Bigg|_{h(r)}^{h_R} = \frac{6\mu U}{m^2} \left(\frac{\delta - 2h_R}{2h_R^2} + \frac{2h-\delta}{2h^2} \right)$$

for δ=0:

$$p(r) - p_\infty = \frac{6\mu U}{m^2} \left(\frac{1}{h} - \frac{1}{h_R} \right)$$

To get the force, we integrate the pressure over the lower plate:

$$-F_z = 2\pi \int_0^R r p(r) dr = \frac{12\pi\mu U}{m^2} \int_0^R \left(\frac{\delta - 2h_R}{2h_R^2} + \frac{2h-\delta}{2h^2} \right) \underbrace{rdr}_{\frac{(h-\delta)}{m^2}dh}$$

$$= \frac{12\pi\mu U}{m^4} \int_\delta^{h_R} \left(\frac{\delta - 2h_R}{2h_R^2}(h-\delta) + \frac{2h^2 - 3\delta h + \delta^2}{2h^2} \right) dh$$

$$= \frac{12\pi\mu U}{m^4} \left(\frac{\delta - 2h_R}{2h_R^2} \left(\frac{1}{2}h^2 - \delta h \right) + h - \frac{3}{2}\delta \ln h - \frac{\delta^2}{2h} \right)\Bigg|_\delta^{h_R}$$

$$= \frac{12\pi\mu U}{m^4} \left[\frac{3}{2}\delta \ln\frac{h_R}{\delta} - \frac{1}{4h_R^2}(h_R - \delta)\left(2h_R^2 + 5h_R\delta - \delta^2\right) \right]$$

Substituting δ=0:

$$\lim_{\delta\to 0}\{F_z\} = \frac{6\pi\mu U h_R}{m^4} \qquad \text{or} \qquad \boxed{\lim_{\delta\to 0}\{F_z\} = O\left(\delta^0\right)}$$

8a.) In the limit of very high Reynolds number, the flow around a submerged object becomes potential flow. In potential flow, the pressure profile is calculated from the velocity profile using Bernoulli's equation:

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{const, say } \frac{p_0}{\rho}$$

where p_0 is the pressure at the stagnation point, where $v^2 = 0$. Solving for p at an arbitrary point:

$$p(r,z) = p_0 - \frac{\rho v^2}{2} = p_0 - \frac{\rho}{2}\left(v_r^2 + v_z^2\right) = p_0 - \frac{\rho}{2}\left[\left(\frac{A}{2}r\right)^2 + (-Az)^2\right]$$

or
$$p(r,z) - p_0 = -A^2 \frac{\rho}{2}\left(\frac{r^2}{4} + z^2\right)$$

Substituting $A \equiv 3\frac{U}{R}$:
$$\boxed{p(r,z) - p_0 = -\frac{9\rho U^2}{2R^2}\left(\frac{r^2}{4} + z^2\right)} \qquad (7)$$

8b.) Prandtl's boundary-layer equation for 2-D flow are

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} - \nu \frac{\partial^2 v_x}{\partial y^2} = -\frac{1}{\rho}\frac{dp_{PF}}{dx}$$

where x is the distance from the stagnation point, measured along the surface and y is the normal distance from the surface. In the homework problems, we obtained the very same equation for the boundary-layer equation for a sphere. In cylindrical coordinates, the distance from the stagnation point is r, while the normal distance from the surface is z. So we make the following substitutions: $x \to r$ and $y \to z$:

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \nu \frac{\partial^2 v_r}{\partial z^2} = -\frac{1}{\rho}\frac{dp_{PF}}{dr}$$

The p_{PF} in this equation is the pressure profile from potential flow, evaluated at the surface z=0. Evaluating (7) at z=0 and differentiating with respect to r:

$$\boxed{v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \nu \frac{\partial^2 v_r}{\partial z^2} = \frac{9U^2}{4R^2} r}$$

The second of Prandtl's boundary-layer equations is just continuity (with no approximations):

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0$$

8c.) The two boundary-layer equations can be reduced to one using the stream function. The single nonlinear PDE for the streamfunction can be solved using a similarity transform.

8d.) The overall problem of uniform flow around a sphere is axisymmetric, with the axis of symmetry passing through the two stagnation points at the north and south poles of the sphere. This symmetry implies that the velocity and pressure profiles should be even functions of θ in spherical coordinates. Even functions have the property that $f(-\theta) = f(+\theta)$ and $df/d\theta = 0$ at $\theta = 0$. In terms of our cylindrical coordinates centered at the forward stagnation point, this implies that $\partial/\partial r = 0$ at r=0. Thus we anticipate that:

$$\frac{d\delta}{dr}=0$$