Final Exam

(Closed Book, 3 hours)

5% 1.) Exactly one of the following boundary-value problems has a boundary layer in its solution $y(x)$ in the limit as $\varepsilon \to 0$:

i.) $y''' + \varepsilon y'' + 2y' = 1$ $\quad y = y' = 0$ at $x = 0$ and $y = 0$ at $x = 1$

ii.) $(1+\varepsilon x)y'' + y = 0$ $\quad y = 1$ at $x = 0$ and $y = \varepsilon$ at $x = 1$

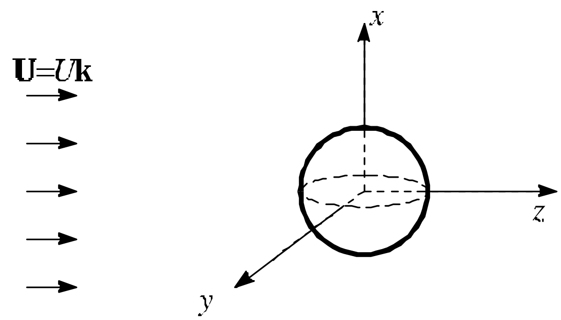
iii.) $\varepsilon y'' - y' = 0$ $\quad y = 0$ at $x = 0$ and $y = 0$ at $x = 1$

iv.) $\varepsilon y'' - y' = 0$ $\quad y = 0$ at $x = 0$ and $y = 1$ at $x = 1$

Which one of the four choices above has the boundary layer in its solution? Explain how you made this choice.

2.) Uniform creeping flow of a viscous fluid at speed U in the z-direction around a rigid sphere of radius R produces a force on the sphere which is given by Stokes law:

$$F_z = 6\pi\mu RU$$



2% a. Under what conditions is "creeping flow" a good approximation. I'm looking for an inequality involving a dimensionless group.

3% b. Describe the "creeping flow approximation." In particular, what has been neglected?

5% c. Give the approximate equations of motion for creeping flow (in terms of velocity **v** and pressure p). These should be written in invariant vector notation. Make sure you have as many equations as unknowns.

2% d. Which of these equations is automatically satisfied by expressing **v** in terms of a stream function ψ?

3% e. How are **v** (or ψ) and p uncoupled in the remaining equation? What mathematical operation is performed?

5% f. What is the partial differential equation which ψ must satisfy in spherical coordinates?

The following parts concern repeating the analysis leading to Stokes equation, replacing the rigid sphere by an inviscid fluid sphere.

10% g. After satisfying the boundary condition at $r\to\infty$, the general solution to Stokes equation for axisymmetric flow around a sphere is

$$\psi(r,\theta) = \left(\frac{c_1}{r} + c_2 r + \frac{U}{2} r^2 \right) \sin^2\theta$$

where $\psi(r,\theta)$ is the streamfunction. Formulate the boundary conditions on the stream-function (you will need two) at $r=R$ which are consistent with an inviscid fluid inside the sphere. <u>You do not need to solve for c_1 and c_2.</u>

2% h. Once ψ and $\mathbf{v}$ are known, how will you determine the pressure profile p? You can write the equation in invariant vector notation.

3% i. For flow around an inviscid sphere, the velocity and pressure profiles turn out to be given by

$$v_r(r,\theta) = U\left(1 - \frac{R}{r}\right)\cos\theta \qquad \text{and} \qquad v_\theta(r,\theta) = -U\left(1 - \frac{R}{2r}\right)\sin\theta$$

$$p(r,\theta) = p_\infty - \mu U R \frac{\cos\theta}{r^2}$$

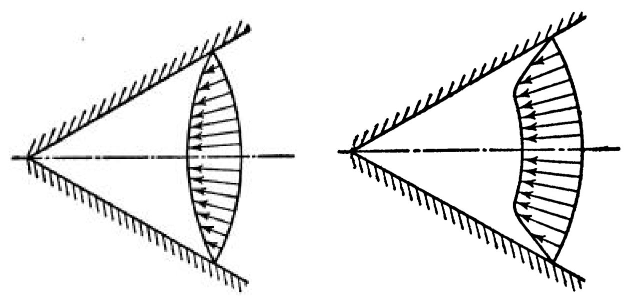
<u>Does this velocity profile satisfy the "no-slip" boundary condition</u>?

10% j. Calculate the force exerted on the sphere.

Hint: it might be helpful to recognize that, for any point on the surface of the sphere, the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ lie in the same plane (which is a ϕ=constant plane). The following relation applies for the unit vectors:

$$\mathbf{e}_z = (\cos\theta)\mathbf{e}_r - (\sin\theta)\mathbf{e}_\theta$$

3.) Consider 2-D flow in a converging channel formed by two flat plates (G. Hamel, 1916). At the interaction of the two plates is a line sink into whicy fluid disappears. Depending on the volumetric flowrate per unit width of plate, Q/W, one of the two velocity profiles shown at right is obtained.

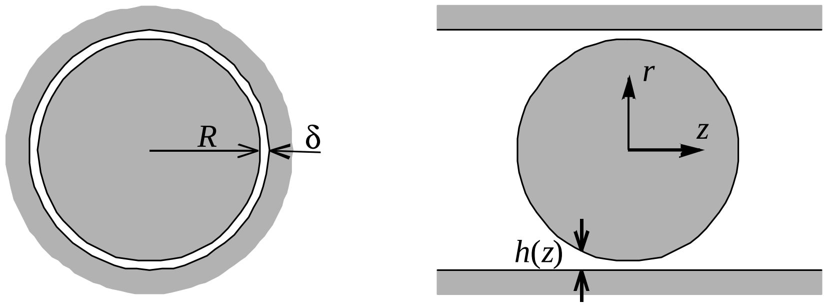


5% a. Which of the two profiles displays a boundary layer?

10% b. Find the solution for the velocity profile which corresponds to <u>potential flow</u>.

10% c. Find the pressure profile corresponding to potential flow.

10% d. Write the differential equations and boundary conditions which describe the velocity profile <u>inside the boundary layer</u>.



15% 4.) A sphere of radius R is located inside a slightly larger tube (tube radius is $R+\delta$). <u>Find the force required to move the sphere along the tube at speed U.</u> The fluid occupying the space between the sphere and the tube wall has viscosity μ and density ρ. Both ends of the tube are open to the atmosphere so the pressure of the fluid outside the gap remains near atmospheric on either side of the sphere.

Hint #1. You may assume that $\delta \ll R$ and that the sphere's center remains on the centerline of the tube. The gap between the sphere and the plate can be approximated (in the vicinity of $z=0$) by

$$h(z) = \delta + \frac{z^2}{2R}$$

which also leads to the following approximations:

$$\int_{-\infty}^{\infty} \frac{dz}{h} = \pi\sqrt{2} \qquad \int_{-\infty}^{\infty} \frac{dz}{h^2} = \frac{\pi\sqrt{2}}{2\delta} \qquad \int_{-\infty}^{\infty} \frac{dz}{h^3} = \frac{3\pi\sqrt{2}}{8\delta^2}$$

Hint #2. The velocity and pressure profiles can more easily be approximated by neglecting the curvature of the tube wall; then the profiles in the narrow gap are essentially the same as in 2-D flow between a cylinder of radius R and a flat plate. In particular, the velocity profile for 2-D flow of the cylinder (whose axis is the z-axis) in the x-direction is a linear combination of linear shear flow and parabolic pressure-driven flow:

$$v_x^{LS} = U\frac{y}{h} \quad \text{and} \quad v_x^{PD} = \frac{1}{2\mu}\frac{dp}{dx}y(y-h)$$

Note: the x,y in the above velocity profiles are <u>not</u> the same x,y as in the figures above.