# Key to Exam #1

1a.)

$$\underline{\underline{\mathbf{S}}}:\underline{\underline{\mathbf{I}}} = S_{xx} + S_{yy} + S_{zz} = 2 + 0 + 0 = 2$$

$$\underline{\underline{\mathbf{E}}} = (\underline{\underline{\mathbf{S}}}:\underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{T}}} = 2\underline{\underline{\mathbf{T}}}$$

$$E_{xx} = 2T_{xx} = 2(5) = \boxed{10}$$

1b.)

$$E_{xz} = 2T_{xz} = 2(-1) = \boxed{-2}$$

1c.)

$$\underline{\underline{\mathbf{T}}}\cdot\mathbf{v} = (5\mathbf{ii}\text{-}\mathbf{ik}+3\mathbf{jk}+2\mathbf{ki})\cdot(4\mathbf{i})$$

$$\underline{\underline{\mathbf{T}}}\cdot\mathbf{v} = 20\mathbf{i}\underbrace{(\mathbf{i}\cdot\mathbf{i})}_{1} - 4\mathbf{i}\underbrace{(\mathbf{k}\cdot\mathbf{i})}_{0} + 12\mathbf{j}\underbrace{(\mathbf{k}\cdot\mathbf{i})}_{0} + 8\mathbf{k}\underbrace{(\mathbf{i}\cdot\mathbf{i})}_{1}$$

$$\underline{\underline{\mathbf{T}}}\cdot\mathbf{v} = \boxed{20\mathbf{i} + 8\mathbf{k}}$$

2a.) For a potential to occur, Theorem III states that $\nabla\times\mathbf{v}$ must vanish <u>everywhere</u>. For linear shear flow:

as $r\to\infty$: $\nabla\times\mathbf{v} = -\Gamma\mathbf{e}_y$

which differs from zero. Thus a **velocity potential does not exist**.

2b.) Flow far from sphere is two-dimensional, but flow near sphere is expected to have $z$-component (not 2D). Flow is neither axisymmetric everywhere nor two-dimensional everwhere. Thus **none of the usual relationships between stream function and velocity apply**.

2c.) Assuming that the sphere acts like a tracer of the fluid, and therefore rotates at the same angular velocity, we estimate its angular velocity as half the curl:

$$\tfrac{1}{2}\left|\nabla\times\mathbf{v}\right| = \boxed{\Gamma/2}$$

2d.) In linear shear flow in the xz-plane, fluid elements rotate around the y-axis. Using the right-hand rule, the direction of the angular velocity is $\boxed{-\mathbf{e}_y}$

3a.) The rate of strain tensor is given by

$$\underline{\underline{\mathbf{d}}} = \tfrac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^t\right] \qquad (1)$$

In linear shear flow, we have only a $z$-component of flow, which depends only on $x$, so $\nabla\mathbf{v}$ has only one non-zero component:

$$\nabla\mathbf{v} = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_i}\mathbf{e}_i\mathbf{e}_j = \frac{\partial v_z}{\partial x}\mathbf{ik} = \Gamma\,\mathbf{ik}$$

taking the transpose: $(\nabla\mathbf{v})^t = \Gamma\,\mathbf{ki}$

substituting into (1): $\boxed{\underline{\underline{\mathbf{d}}} = \tfrac{1}{2}\Gamma(\mathbf{ik}+\mathbf{ki})}$

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3b.) The viscous stress tensor is given by Newton's law of viscosity:

$$\underline{\underline{\tau}} = 2\mu\underline{\underline{\mathbf{d}}} = \boxed{\Gamma(\mathbf{ik}+\mathbf{ki})}$$

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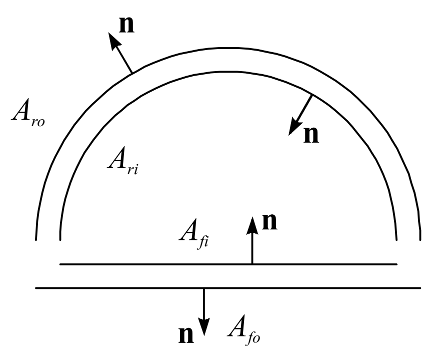
4.) The pressure <u>inside</u> the hut will equilibrate to the local value of pressure <u>outside</u> the opening at θ. So we first decide what pressure inside the hut will create a downward force which will offset the upward lift. we are looking for the net force on the roof of the hut:




$$\mathbf{F}_{roof} = -\int_{A_{roof}}\mathbf{n}p\,da = \underbrace{-\int_{A_{ro}}\mathbf{n}p\,da}_{\mathbf{F}_{ro}}\underbrace{-\int_{A_{ri}}\mathbf{n}p_s\,da}_{\mathbf{F}_{ri}} \qquad (2)$$

Notice that we have decomposed the force on the roof into an inner and an outer contribution. There are two methods to evaluate the force on the outside of the roof: 1) direct evaluation of the surface integral using the known pressure profile, or 2) use the result from our homework solution:

$$\mathbf{F}_p = -\int_{A_{hwk}}\mathbf{n}p\,da = \underbrace{-\int_{A_{ro}}\mathbf{n}p\,da}_{\mathbf{F}_{ro}}\underbrace{-\int_{A_{fo}}\mathbf{n}p\,da}_{\mathbf{F}_{fo}} = \tfrac{5}{3}\rho U^2 LR\mathbf{e}_y \qquad (3)$$

<u>Method #1</u>:

$$\mathbf{F}_{ro} = -\int_{A_{ro}}\mathbf{n}p\,da \qquad (4)$$

The pressure profile on the outer surface of the roof is given by:

$$p(R,\theta) = p_0 + \tfrac{1}{2}\rho V^2\left(1-4\sin^2\theta\right) \qquad (5)$$

while the unit normal can be decomposed into Cartesian components according to

$$\mathbf{n} = (\cos\theta)\mathbf{e}_x + (\sin\theta)\mathbf{e}_y$$

We expect the net force to act solely in the $y$-direction, so we dot both sides of (4) by $\mathbf{e}_y$:

$$F_{ro,y} = -\int_0^\pi \underbrace{\left(\mathbf{n}\cdot\mathbf{e}_y\right)}_{\sin\theta} p(\theta)\,LR\,d\theta = -LR\int_0^\pi \left[p_0 + \tfrac{1}{2}\rho V^2 \underbrace{\left(1-4\sin^2\theta\right)}_{4\cos^2\theta-3}\right] \underbrace{\sin\theta\,d\theta}_{-d\cos\theta\equiv -du}$$

$$= LRp_0 \underbrace{\int_1^{-1} du}_{-2} + \tfrac{1}{2}\rho V^2 LR \underbrace{\int_1^{-1}\left(4u^2-3\right)du}_{\underbrace{\left(\frac{4}{3}u^3\right)\Big|_1^{-1}}_{-\frac{8}{3}} \underbrace{-\left(3u\right)\Big|_1^{-1}}_{6}} = -2LRp_0 + \tfrac{5}{3}\rho V^2 LR$$

<u>Method #2</u>: This integration can be avoided by making use of (3):

$$-\int_{A_{ro}} \mathbf{n}p\,da = \tfrac{5}{3}\rho U^2 LR\mathbf{e}_y + \underbrace{\int_{A_{fo}} \mathbf{n}p\,da}_{\left(-\mathbf{e}_y\right)p_0 \underbrace{A_{fo}}_{2LR}} = \left(-2p_0 + \tfrac{5}{3}\rho U^2\right)LR\mathbf{e}_y \tag{6}$$

The pressure on the bottom of the half cylinder (in the homework problem) equals $p_0$ (the pressure in the fluid far from the cylinder). Note that the result from (6) is identical to that from Method #1.

In addition, we know the net force on the hut caused by a uniform pressure $p_s$ inside vanishes:

$$\underbrace{-\int_{A_{ri}} \mathbf{n}p_s\,da}_{\mathbf{F}_{ri}} \underbrace{-\int_{A_{fi}} \mathbf{n}p_s\,da}_{\mathbf{F}_{fi}} = \mathbf{0}$$

We can use this result to evaluate $\mathbf{F}_{ri}$ without integrating over the curved surface:

$$\underbrace{-\int_{A_{ri}} \mathbf{n}p_s\,da}_{\mathbf{F}_{ri}} = \underbrace{\int_{A_{fi}} \mathbf{n}p_s\,da}_{\left(\mathbf{e}_y\right)p_s \underbrace{A_{fi}}_{2RL}} = 2RLp_s\mathbf{e}_y \tag{7}$$

(6) and (7) into (2):

$$\mathbf{F}_{roof} = \underbrace{\left(-2p_0 + \tfrac{5}{3}\rho V^2\right)LR\mathbf{e}_y}_{\mathbf{F}_{ro}} + \underbrace{2RLp_s\mathbf{e}_y}_{\mathbf{F}_{ri}} = \left[2\left(p_s - p_0\right) + \tfrac{5}{3}\rho V^2\right]LR\mathbf{e}_y$$

Setting this equal to **0** and solving for $p_s$:

$$p_s = p_0 - \tfrac{5}{6}\rho V^2 \tag{8}$$

Now the question which remains is "at what θ is the outside pressure equal to this value?" Comparing (8) to (5):

$$p_s = p_0 - \tfrac{5}{6}\rho V^2 = p_0 + \tfrac{1}{2}\rho V^2\left(1 - 4\sin^2\theta\right)$$

$$-\tfrac{5}{6} = \tfrac{1}{2}\left(1 - 4\sin^2\theta\right)$$

$$4\sin^2\theta - 1 = \tfrac{5}{3}$$

$\sin^2\theta = \tfrac{2}{3}$ or **$\theta = 54.7°$** or **$\theta = 0.995$ radians**