Chemical Engineering Mathematics 06-262
Homework 9
Due: Thursday April 19, 2001

Differential (Shell) Balance Problems

1. **(15 points)** You are asked to help design the insulation system for a spherical liquid air tank. The 3 m diameter tank is full of liquid air and at a **constant** temperature of -150 ^{O}C. The tank is covered in a 50 cm thick layer of solid insulation.

    a) Develop an ODE to describe the steady state temperature profile in the insulation. Use a shell balance to develop the expression and assume that the flux is given by Fourier's law, $q = -k\ dT/dr$, where $k = 0.03$ W/m^{o}C.
    b) Solve the ODE to determine the general expression for the temperature in the insulation.
    c) Find the particular solution of the expression using the following conditions:

        ⇒ the temperature at the inner surface of the insulation ($R_i = 1.5$ m) is the temperature of liquid air ($T = -150\ ^{o}C$)

        ⇒ at the outer surface of the insulation, energy escapes such that the temperature gradient is proportional to the difference in temperature between the surface and the ambient air which is at 18^{o}C:

        $$\frac{dT}{dr} \text{ at surface} = -h\ \{T(R_o) - 18^{o}C\}$$

        the constant is $h = 0.01\ cm^{-1}$.
    d) What is the temperature at the outer wall? As an engineer, does this analysis help you choose an insulating material (i.e., can you determine the best value of k)?

2. **(15 points)** Red blood cells release oxygen in the capillaries, where the dissolved gas diffuses into the surrounding tissue. Consider a capillary to be a well-mixed tube of radius R_o surrounded by a concentric cylinder of tissue of radius R_1 ($R_1 > R_o$). The concentration of oxygen at the wall of the capillary (and everywhere inside the capillary) is C_o mole/L and the flux into the tissue at the inner wall (dc/dr at $r = R_o$) is known to be 2 mol/Lm. The tissue is assumed to consume oxygen at a rate of M mole/sec-L. We need to derive an expression for the steady-state concentration of oxygen as a function of radius in the tissue.

    a) Perform a radial shell balance on the tissue, assuming that the flux of oxygen obeys Fick's law: $q = -D\ dc/dr$ where D is a material constant (units of m^2/s).
    b) Write out the ODE that describes the radial concentration of oxygen. The values of the system parameters are $C_o = 3$ mol/L, $D = 2\ m^2/sec$, $M = 1$ mol/sec-L. The inner radius of the capillary is 1mm and the tissue is 1mm thick.
    c) Write the boundary conditions in terms of r. Apply these boundary conditions to determine the particular solution.

d) What is the concentration of oxygen at the outer surface of the tissue? What fraction of the oxygen is fixed (i.e., reacted away) by the tissue of the capillary?

Laplace Transforms

3. **(15 points)** Find the Laplace transforms or inverse Laplace transforms of the following functions. Use the tables in the text or passed out in class. If necessary, use the definition of the Laplace transform to integrate and solve.

   a) $y(t) = (1 + \exp(2t))^2$
   b) $y(t) = \cos(a\,t + b)$
   c) $y(t) = \int (t^2 + t)\,dt$
   d) $Y(s) = \dfrac{6s^2 + 50}{(s + 3)(s^2 + 4)}$
   e) $Y(s) = \dfrac{4s}{4s^2 + 1}$

4. **(15 points)** For each of the following nonhomogeneous differential equations, take the Laplace transform of the differential equations, manipulate the algebraic expression in Laplace space and invert Y(s) to find the particular solution to the differential equation y(t).
   a) $h' + 2\,h = e^{-4t}$; $h(0) = 10$
   b) $y'' + 4\,y' + 4\,y = 4$; $y(0) = 0$, $y'(0) = 4$
   c) $y'' + 25\,y = \int (2t^2 - \exp(-4t))\,dt$; $y(0) = 3$, $y'(0) = 2$ **see pg 262 in text.

   Note that the partial fractions in this part are painful; feel free to use MathCAD's symbolic features to find the partial fractions of complicated terms, although you can do them by hand if you want.

Laplace Transforms – Unit Step Function

5. **(15 points)** Convert the following functions into single-line expressions using combinations of unit step functions. For each case, plot the function versus time. *Note: MathCAD uses Φ(t) to denote the unit step function, upper case F-ctrl-G.*

   a. $g(t) = \begin{cases} 0 & t < 10 \\ t + 10 & t > 10 \end{cases}$

   b. $g(t) = \begin{cases} 1 & t < 10 \\ 5 & 10 < t < 20 \\ 2 & t > 20 \end{cases}$

c. $g(t) = \begin{cases} 0 & t < -2 \\ 1 & -2 < t < 2\pi \\ 1 + \sin(t) & 2\pi < t < 8\pi \\ 1 & t > 8\pi \end{cases}$

Laplace Transforms – Coupled Systems of Equations

6. **(15 points)** Solve the following coupled systems of first order differential equations ($\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$) by: writing the system as two equations, taking the Laplace transform of each, solving for the functions $Y_1(s)$ and $Y_2(s)$, then inverting to find $y_1(t)$ and $y_2(t)$.
7. $\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ and $\mathbf{g} = \mathbf{0}$; $\mathbf{y_0} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$
8. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{g} = \begin{bmatrix} 2[1 - u(t-2\pi)] \cos(t) \\ 0 \end{bmatrix}$; $\mathbf{y_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$