Chemical Engineering Mathematics 06-262
Homework 8
Due: Friday April 13, 2001 by noon

***For several of these problems, you will note that you solved the homogeneous part in homework #7. You do not need to repeat this step, feel free to get the homogeneous solution from the homework solutions if applicable.*

Coupled Systems of First Order Differential Equations - Nonhomogeneous

1. **(15 points)** For the following set of coupled, first order differential equations with the initial condition that $y_1(0) = 1$ and $y_2(0) = 0$:

$$y_1' = -y_1 + y_2 + f(x)$$

$$y_2' = -2y_1 + y_2 + h(x)$$

   a. If $f(x) = 2$ and $h(x) = 0$ find the general and particular solution for $\mathbf{y}(t)$. Plot $y_1(t)$ versus $y_2(t)$ and comment on the form of the solution space.
   b. If $f(x) = 0$ and $h(x) = 2x + 1$ find the general and particular solution for $\mathbf{y}(t)$. Plot $y_1(t)$ versus $y_2(t)$ and comment on the form of the solution space.
   c. If $f(x) = \sin(2x)$ and $h(x) = 5x^2$ find the general and particular solution for $\mathbf{y}(t)$. Plot $y_1(t)$ versus $y_2(t)$ and comment on the form of the solution space.

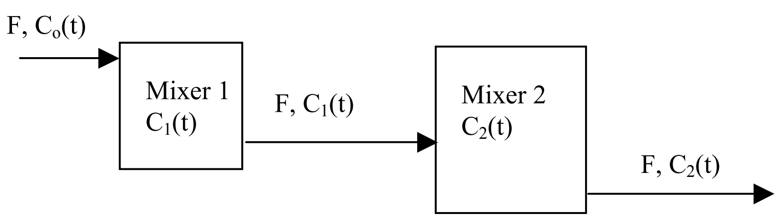
2. **(15 points)** Given the following linear third order differential equation with initial conditions:

$$y^{(3)} + 2y'' + 9y' + 18y = 4e^{-t}\sin(2t); \quad y(0) = 1, y'(0) = 0, y''(0) = 0$$

   a. Write this single third order ODE as a system of three coupled FODE.
   b. Solve for the general solution for the three unknown functions $y_1(t)$, $y_2(t)$ and $y_3(t)$.
   c. Find the particular solution for each function and plot $y_1(t)$, $y_2(t)$ and $y_3(t)$.

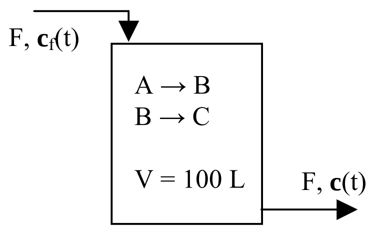
Modeling of Nonhomogeneous Coupled First Order Systems

3. **(15 points)** A mixing system is set up to dilute the product of a reaction to different levels for sale to a variety of customers. Two mixing vessels (tanks) are attached and material flows through the reactor at a constant volumetric flow rate, F. The feed to the system is concentrated solution, $C_o(t) = 10$ gm A/m^3. The second tank is initially filled with solvent ($C_2(0) = 0$) while the first tank is initially filled with a concentrated solution $C_1(0) = 5$ gm A/m^3, and the two vessels have different volumes.



a. Develop a system of coupled equations that describes the rate of change of concentration of A in each of the vessels.
b. For the case that $V_1 = 50\ m^3$, $V_2 = 200\ m^3$ and $F = 10\ m^3/min$, solve the system of coupled equations for the concentration of A as a function of time.
c. Plot the concentration of A in each tank as a function of time. How long does it take for the system to reach steady state? If you want to sell product at a concentration of 3 gm A/m^3, at what time do you stop the flow, cap mixer 2 and sell the product?

4. **(15 points)** You are designing a continuous stirred tank reactor (CSTR). In the well-mixed reactor, A reacts to form the intermediate B at a rate proportional to the concentration of A in the reactor. Simultaneously, B reacts to form the product C at a rate proportional to the concentration of B.

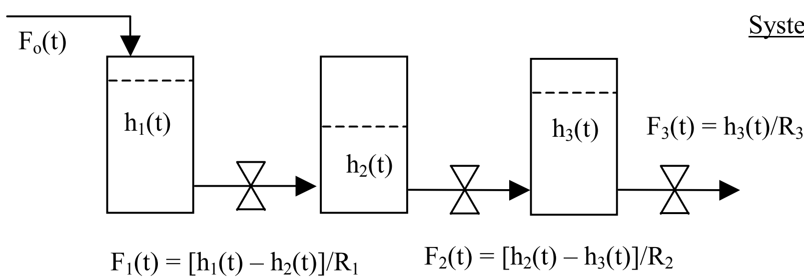


Rate of A consumed in the reaction $A \rightarrow B$ is $k_1\, C_A(t)$ where $k_1 = 0.5\ min^{-1}$.

Rate of B consumed in the reaction $B \rightarrow C$ is $k_2\, C_B(t)$ where $k_2 = 0.3\ min^{-1}$.

a. Perform separate balances on each component and develop models for the rate of change of concentration of each component (C_A, C_B, and C_C). The feed is 0.5 mol/L A and 0.5 mol/L C and flows at a rate of 10 L/min.
b. Solve the coupled system of equations for the concentrations as a function of time. Write the general solution for the unknown concentration vector $\mathbf{c} = [C_A\ \ C_B\ \ C_C]^T$.
c. If the reactor is initially filled with solvent and A (i.e., $C_A(t=0) = 1$ mol/L and $C_B(t=0) = C_B(t=0) = 0$ mol/L), plot the concentrations of A, B and C in the reactor as a function of time. How long does the system take to reach steady state?

5. **(15 points)** Three tanks are attached in series as shown below. All three tanks have the same constant cross sectional area.



System Data:
$A = 5\ m^2$
$R_1 = 2\ hr/m^2$
$R_2 = 1\ hr/m^2$
$R_3 = 1\ hr/m^2$
$F_o(t) = 10 m^3/hr\ (1-e^{(-t/\tau)})$
$\tau = 1$ hr

a. Perform mass balances on each of the three tanks and develop a model for the rate of change of height of liquid in each tank. Write these three mass balances as a matrix equation **h'** = **A h + g** and identify all terms in the vector **g** and matrix **A**.
b. Solve the system of coupled equations for **h**(t).
c. If the first and third tanks are initially full (h = 10 m) and the middle tank is initially empty, plot the heights of each tank as a function of time.
d. At what time does the middle tank reach its maximum height?

Numerical Methods for solving Coupled Systems of FODE

6. (**15 points**) You will learn in kinetics that many rate equations do not have simple linear dependence on concentration; this often gives rise to nonlinear terms in the mass balance equations. In problem 4, the reactions might be such that the rate of A consumed is proportional to the $C_A(t)^2$ and the rate of B consumed is proportional to $C_B(t)^2$. In this case, $k_1 = 0.5$ Lmol^{-1}min^{-1} and $k_2 = 0.3$ L mol^{-1}min^{-1}.
   a. Perform balances on each component and develop models for the rate of change of concentration of each component (C_A, C_B, and C_C) assuming F = 0 L/min.
   b. Solve the coupled system of equations for the concentrations as a function of time if the reactor is initially filled with solvent and A (i.e., $C_A(t{=}0)$ =10 mol/L and $C_B(t{=}0) = C_B(t{=}0) = 0$ mol/L), plot the concentrations of A, B and C in the reactor as a function of time.
   c. How long does the system take to reach steady state?