Chemical Engineering Mathematics 06-262
Homework 7
Due: Thursday April 5, 2001 (red text denotes corrections 3/26/01)

***For problems 1 & 2, calculate eigenvectors and eigenvalues by hand. For all other problems, feel free to use MathCAD or your calculator.*

Coupled Systems of First Order Differential Equations

1. **(15 points)** For the following set of coupled, first order differential equations with the initial condition that $y_1(0) = 1$ and $y_2(0) = 0$:

$$y_1' = a\,y_1 + b\,y_2$$
$$y_2' = -2\,y_1 + y_2$$

   a. If a = 0 and b = -1, find the general and particular solution for **y**(t). Plot $y_1(t)$ versus $y_2(t)$ and comment on the form of the solution space.
   b. If a = 5 and b = 2, find the general and particular solution for **y**(t). Plot $y_1(t)$ versus $y_2(t)$ and comment on the form of the solution space.
   c. If a = -1 and b = 1, find the general and particular solution for **y**(t). Plot $y_1(t)$ versus $y_2(t)$ and comment on the form of the solution space.

2. **(15 points)** For the following homogeneous second order differential equation:

$$y'' + 2\,y' + y = 0;\;\; y(0) = 1,\, y'(0) = 0$$

   a. Solve this as a constant coefficient, second order equation and find the general and particular solutions for y(t).
   b. Using the substitutions $y_1(t) = y(t)$ and $y_2(t) = y'_1(t)$, rewrite the *second* order equation as a system of two coupled *first* order equations (see pg. 156 Kreyszig).
   c. Solve the system in part (b) for the general and particular solutions of **y**. Write the particular solutions for $y_1(t)$ and $y_2(t)$.
   d. Are the solutions from parts (a) and (b) the same?

3. **(15 points)** Given the following linear third order differential equation with initial conditions:

$$y^{(3)} + 2\,y'' + 9\,y' + 18\,y = 0; \qquad y(0) = 1,\, y'(0) = 0,\, y''(0) = 0$$

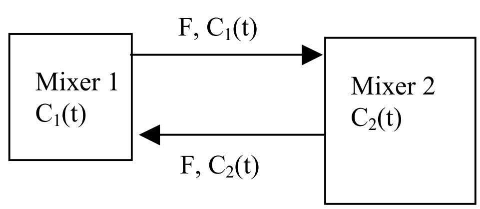
   where $y^{(3)}$ denotes the third derivative of the unknown function y(x).

   a. Convert this single third order system into a system of three coupled first order differential equations.

b. Solve for the general solution for the three unknown functions $y_1(t)$, $y_2(t)$ and $y_3(t)$.
c. Find the particular solution for each function and plot $y_1(t)$, $y_2(t)$ and $y_3(t)$.
d. Plot the particular solution for the unknown function $y(t)$.

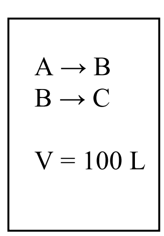
Modeling of Coupled First Order Systems

4. **(15 points)** A mixing system is set up to dilute the product of a reaction to different levels for sale to a variety of customers. Two mixing vessels (tanks) are attached and a mixture of solvent plus product (call it A) flows through the system at a constant volumetric flow rate, F. The mixture in vessel 1 flows into vessel 2 and then back into vessel 1. The second tank is initially filled with solvent ($C_2(0) = 0$) while the first tank is initially filled with a concentrated solution $C_1(0) = 10$ gm A/m^3, and the two vessels have different volumes.



a. Develop a system of coupled equations that describes the rate of change of concentration of A in each of the vessels.
b. For the case that $V_1 = 50$ m^3, $V_2 = 200$ m^3 and $F = 10$ m^3/min, solve the system of coupled equations for the concentration of A as a function of time.
c. Plot the concentration of A in each tank as a function of time. How long does it take for the system to reach steady state? If you want to sell product at a concentration of 3 gm A/m^3, at what time do you stop the mixing, cap mixer 2 and sell the product?

5. **(15 points)** You are designing a batch reactor (see picture). In the well-mixed reactor, A reacts to form the intermediate B at a rate proportional to the concentration of A in the reactor. Simultaneously, B reacts to form the product C at a rate proportional to the concentration of B.



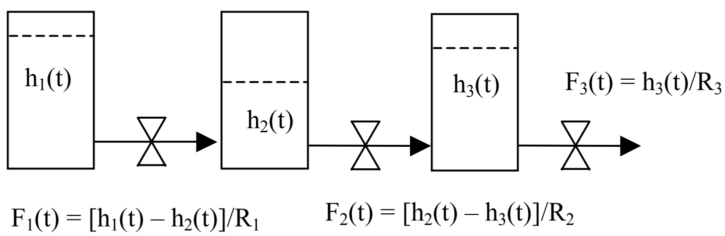
Rate of A consumed in the reaction $A \rightarrow B$ is $k_1 C_A(t)$ where $k_1 = 0.5$ min^{-1}.

Rate of B consumed in the reaction $B \rightarrow C$ is $k_2 C_B(t)$ where $k_2 = 0.3$ min^{-1}.

a. Perform separate balances on each component and develop models for the rate of change of concentration of each component (C_A, C_B, and C_C).

b. Solve the coupled system of equations for the concentrations as a function of time. Write the general solution for the unknown concentration vector **c** = $[C_A \;\; C_B \;\; C_C]^T$.
c. If the reactor is initially filled with solvent and A (i.e., $C_A(t=0) = 10$ mol/L and $C_B(t=0) = C_B(t=0) = 0$ mol/L), plot the concentrations of A, B and C in the reactor as a function of time. How long does the system take to reach steady state?

6. **(15 points)** Three tanks are attached in series as shown below. All three tanks have the same constant cross sectional area.



System Data:
$A = 5$ m^2
$R_1 = 2$ hr/m^2
$R_2 = 1$ hr/m^2
$R_3 = 1$ hr/m^2

a. Perform mass balances on each of the three tanks and develop a model for the rate of change of height of liquid in each tank.
b. Write these three mass balances as a matrix equation **h**' = **A h** and identify all terms in the vector **h** and matrix **A**.
c. Solve the system of coupled equations for **h**(t).
d. If the first and third tanks are initially full (h = 10 m) and the middle tank is initially empty, plot the heights of each tank as a function of time.
e. At what time does the middle tank reach its maximum height?