Mathematical Methods of Chemical Engineering 06-262
Homework 6
Due: Wednesday March 7, 2001 (I will accept homework until noon on 3/8/01)

First Order Differential Equations – form of solutions

1. **(12 points)** Download the MathCAD document *feb22_note.mcd* from the course web site. To get a feel for the types of solutions possible from a first order system with forcing functions, answer the following. Start by setting the process variables to: V = 10 m^3, F = 10 m^3/hr, k = 0.5 hr^{-1}, Cinit = 10 kg/m^3 and cf = 10 kg/m^3.
   a. For the tank draining with a constant feed composition, what does the inlet feed composition (cf) need to be to keep the reactor concentration constant at all times?
   b. Change the feed composition (cf) to 30 kg/m^3. For the tank with a slowly increasing feed composition, what is the maximum concentration of ethyl acetate seen in the tank? At what time does this maximum occur?
   c. For a feed composition of 30 kg/m^3, what can you do to the feed profile in the discontinuous feed situation (case 4), if the concentration in the tank must be kept below 18 kg/m^3?
   d. In the case of the reactor with an oscillating feed composition, should the concentration in the tank be in-phase (i.e., max at the same time) or out-of-phase with the inlet composition?

Homogeneous, Linear 2nd Order ODEs with Constant Coefficients

2. **(12 points)** Solve the following second order differential equations. Provide a general and particular solution.
   a. $\theta'' - 2\theta' + 10\theta = 0$; $\theta(0) = 2$, $\theta(\pi/2) = -\exp(\pi/2)$
   b. $z'' + 4z = 0$; $z(0) = 0$, $z'(\pi) = 2$

Homogeneous, Linear 2nd Order ODEs with Non-constant Coefficients

3. **(12 points)** A class of second order linear differential equations are known as Euler-Cauchy equations and are of the form:

$$x^2 y'' + a x y' + b y = 0.$$

   These equations do **not** have constant coefficients. However, they are of a form that lends itself to a set of basis functions and solution similar to the technique used for constant coefficient problems.
   a. Assume that a valid solution to the ODE is $y(x) = x^m$. Show that this is a solution for certain values of *m*. Which values of *m* that are solutions to the equation?
   b. State the general solution and basis functions. Show that the basis functions are linearly independent and for which cases linear independence is lost.

c. Find the general and particular solutions to
$$x^2y'' + 6xy' + 4y = 0;\ y(1) = 3,\ y'(1) = 3.$$
d. Find the general and particular solutions to
$$y'' - (3/x)\,y' + (4/x^2)y = 0;\ y(1) = 5,\ y'(1) = 12.$$

<u>Setting up and Solving First Order Differential Systems</u>

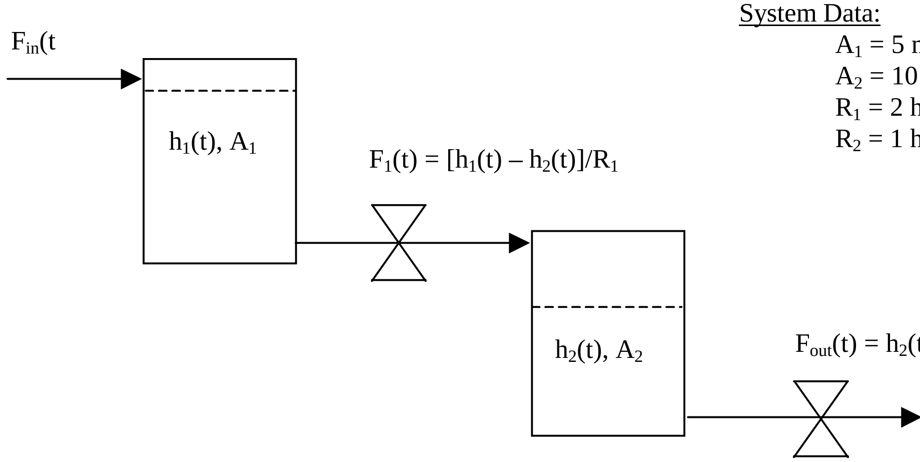
4. **(12 points)** The following are first order systems, meaning that their dynamics are described by ordinary first order differential equations. For each physical problem, develop a differential equation that describes the system, state which type of ODE best describes the model (i.e., linear, homogeneous, separable etc…) and state which method(s) could be used to solve the ODE. State any assumptions that you make and state the units of all variables that you define. *You do not need to solve the ODEs, just set up a model.*
   a. Liquid nitrogen is stored in a spherical tank. The temperature of the liquid N_2 is T_o when delivered. It is safe to assume that neither the heat capacity of the system (tank + N_2), C_p, nor the density of the N_2, ρ, change with temperature. Heat is lost through the surface of the tank, such that the rate of heat loss is proportional to the difference between the tank temperature and the ambient temperature $(T - T_{amb})$ and the surface area of the tank. A cooling system removes heat at a rate of $Q(t)$ given in watts, but is malfunctioning so that it fluctuates $Q(t) = Q_o(1 + \cos(\omega t))$. How does the temperature change with time?
   b. A cylindrical tank with constant cross section, A, is being drained through a resistance valve such that flow rate out is proportional to the height of liquid in the tank. Liquid is flowing into the tank through a control valve with flow rate $F_{in}(t)$. This valve responds to the volume of liquid in the tank and has the form $F_{in}(t) = \alpha h(t) + \sin(\omega t)$. If the tank is initially filled to a level of h_0, what is the height of liquid in the tank as a function of time?

<u>Second Order Nonhomogeneous Differential Equations –Undetermined Coefficients</u>

5. **(12 points)** For the following linear second order differential equations, find the solution to the homogeneous ODE. Then use the method of Undetermined Coefficients to find the particular solution. State the general and particular solution to the ODE.
   a. $y'' + y' - 6y = 28e^{4x};\ y(0) = 2,\ y'(0) = 3$
   b. $z'' + 2z' + 5z = 4e^{-x} - 185\sin(4x);\ z(0) = 1/2,\ z'(0) = 17/2$
   c. $h'' + 2.2h' + 0.2h = 5(t + e^{-2t});\ h(0) = 1,\ h'(0) = 1$
   d. $c'' + 6c' + 9c = -2e^{-3x};\ c(0) = 0,\ c'(0) = 5$

Second Order Homogeneous Differential Equations

6. **(12 points)** We have dealt with one tank draining through a valve and ended up with a first order system. Two tanks in series have second order dynamics. For example, given two tanks that are connected as shown below, where the feed to tank 1 is a function only of time, $F_{in}(t)$. Both tanks are cylindrical, but have different cross sectional areas given by A_1 and A_2. Water flows out of tank 2 via a resistance valve such that the flow rate is proportional to the height of liquid in tank 2, $h_2(t)$. The system becomes second order because the flow rate through the valve between the two tanks is proportional to the *difference* in heights of water in the two tanks.



System Data:
$A_1 = 5$ m^2
$A_2 = 10$ m^2
$R_1 = 2$ hr/m^2
$R_2 = 1$ hr/m^2

a. Perform a mass balance on tank 1 to develop an expression for the change in height in tank 1 as a function of time, dh_1/dt. Perform a similar balance on tank 2 and develop an expression for dh_2/dt. Keep these in terms of the variables in the problem (A_1, A_2, R_1, R_2 and $F_{in}(t)$)
b. Eliminate $h_2(t)$ in these two equations. Do this by solving the balance on tank 1 for $h_2(t)$. Substitute the expression for h_2 (t) and the derivative of this equation into the balance from tank 2. You should now have a linear, second order differential equation for $h_1(t)$: $h_1'' + a h_1' + bh_1 = r(t)$. Evaluate *a* and *b*.
c. For the homogeneous case (i.e., $F_{in}(t) = 0$), find the general solution for $h_1(t)$. Find the particular solution for $h_1(t)$ if $h_1(0) = 5$ m and $h_2(0) = 10$ m. Plot $h_1(t)$ and $h_2(t)$ for this case.