1. **The following third order differential equation describes the inertial response of the cantilever in an atomic force microscope:**

$$y^{(3)} + 2y'' + 9y' + 18y = 18(2t - 1)\;;\; y(0) = 0,\; y'(0) = 1,\; y''(0) = 1$$

a) **(5 points)** Convert this third order equation in y(x) to a system of coupled first order differential equations. Identify all elements in the matrices and vectors.
b) **(8 points)** The eigenvalues for this coupled system are -2, 3i and -3i. Find the general solution to the homogeneous coupled system.
c) **(7 points)** Solve for the solution of the nonhomogeneous, coupled system using the method of Undetermined Coefficients.
d) **(5 points)** Use the initial conditions to solve for the arbitrary constants.

$$\left.\begin{array}{l} y_1 \equiv y \\ y_2 \equiv y' \\ y_3 \equiv y'' \end{array}\right\} \quad \left.\begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -2y_3 - 9y_2 - 18y + 18(2t-1) \end{array}\right\}$$

** Matrices defined below

Find $\underline{x}$'s for $\lambda_1 = -2 \quad \lambda_2 = 3i \quad \lambda_3 = -3i$

$$\lambda_1 = -2 \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -18 & -9 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0} \Rightarrow \left.\begin{array}{l} 2x_1 = -x_2 \\ 2x_2 = -x_3 \end{array}\right\} \rightarrow \underline{x}^{(1)} = \begin{bmatrix} -\frac{1}{2} & 1 & -2 \end{bmatrix}^T$$

$$\lambda_2 = 3i \quad \begin{bmatrix} -3i & 1 & 0 \\ 0 & -3i & 1 \\ -18 & -9 & -2-3i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0} \Rightarrow \left.\begin{array}{l} -3i\,x_1 = -x_2 \\ -3i\,x_2 = -x_3 \end{array}\right\} \rightarrow \underline{x}^{(2)} = \begin{bmatrix} 1 & 3i & -9 \end{bmatrix}^T$$

$$\lambda_3 = -3i \quad \begin{bmatrix} 3i & & 0 \\ 0 & 3i & 1 \\ -18 & -9 & -2+3i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0} \Rightarrow \left.\begin{array}{l} 3i\,x_1 = -x_2 \\ 3i\,x_2 = -x_3 \end{array}\right\} \rightarrow \underline{x}^{(3)} = \begin{bmatrix} 1 & -3i & -9 \end{bmatrix}^T$$

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$$y_h(x) = c_1 \underline{x}^{(1)} e^{-2t} + c_2 \underline{x}^{(2)} e^{-3it} + c_3 \underline{x}^{(3)} e^{-3it}$$

$$** \; \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & -9 & -2 \end{bmatrix} \text{ and } \mathbf{g} = 18\begin{bmatrix} 0 \\ 0 \\ 2t - 1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Problem 1 (con't) $\begin{bmatrix}-1\\3i\\9\end{bmatrix} e^{-3it} = \begin{bmatrix}-1\\3i\\9\end{bmatrix}\left\{\cos(3t)+i\sin(3t)\right\} = \begin{bmatrix}-\cos(3t)\\-3\sin(3t)\\9\cos(3t)\end{bmatrix} + i\begin{bmatrix}-\sin(3t)\\3\cos(3t)\\9\sin(3t)\end{bmatrix}$

$$\boxed{y_h(x) = C_1\begin{bmatrix}1\\-2\\4\end{bmatrix}e^{-2t} + C_2\begin{bmatrix}-\cos(3t)\\-3\sin(3t)\\9\cos(3t)\end{bmatrix} + C_4\begin{bmatrix}-\sin(3t)\\3\cos(3t)\\9\sin(3t)\end{bmatrix}}$$

Nonhomogeneous: assume $\underline{y}_p = \underline{u} + \underline{v}t \quad \therefore \underline{y}_p' = \underline{v}$

$\therefore \quad \underline{v} = \underline{A}(\underline{u} + \underline{v}t) + \underline{g}$ where $\underline{g} = \begin{bmatrix}0\\0\\36\end{bmatrix}t + \begin{bmatrix}0\\0\\-18\end{bmatrix}$

$\Rightarrow$ t terms: $-\underline{\underline{A}}\,\underline{v} = \begin{bmatrix}0\\0\\36\end{bmatrix} \Rightarrow \left.\begin{matrix}-v_2 = 0\\ -v_3 = 0\\ -(-18v_1 - 9v_2 - 2v_3) = 36\end{matrix}\right\}$ $\begin{matrix}v_2 = v_3 = 0\\ v_1 = 2\end{matrix}$

$\Rightarrow$ t^0 terms: $\underline{v} - \underline{\underline{A}}\,\underline{u} = \begin{bmatrix}0\\0\\-18\end{bmatrix} \Rightarrow \left.\begin{matrix}2 - u_2 = 0\\ 0 - u_3 = 0\\ 18u_1 + 9u_2 + 2u_3 = -18\end{matrix}\right\}$ $\begin{matrix}u_2 = 2\\ u_3 = 0\\ u_1 = -2\end{matrix}$

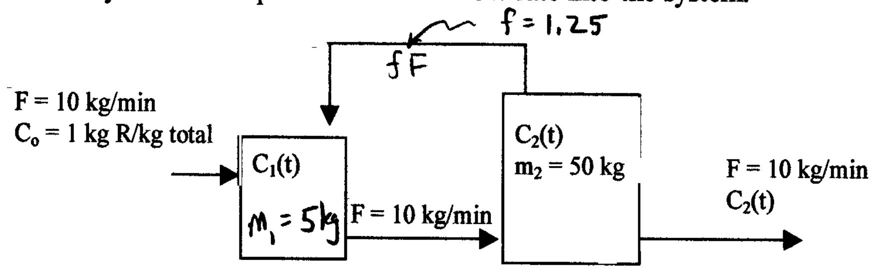
$$\boxed{\underline{y}_p = \begin{bmatrix}-2\\2\\0\end{bmatrix} + \begin{bmatrix}2\\0\\0\end{bmatrix}t}$$

$\underline{y} = \underline{y}_h + \underline{y}_p$ @ $t = 0$ $\quad \underline{y} = \begin{bmatrix}0\\1\\1\end{bmatrix}$

$\therefore \underline{y}(0) = C_1\begin{bmatrix}1\\-2\\4\end{bmatrix} + C_2\begin{bmatrix}-1\\0\\9\end{bmatrix} + C_3\begin{bmatrix}0\\3\\0\end{bmatrix} + \begin{bmatrix}-2\\2\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$

$\left.\begin{matrix}c_1 - c_2 - 2 = 0\\ -2c_1 + 3c_3 + 2 = 1\\ 4c_1 + 9c_2 = 1\end{matrix}\right\} \longrightarrow \quad c_1 = \frac{19}{13} \quad c_2 = \frac{-7}{13} \quad c_3 = \frac{25}{13}$

2. Reaction products are often concentrated in a stage-wise manner to avoid the dangers of product overheating caused by large exothermic heats of dilution. A hot reaction product is pumped into a sequence of two vessels (also known as tanks). The purpose of the system is to concentrate the reaction product R solution. The first vessel is 10 times smaller than the second and reaction product is recycled within the two vessels. The recycle rate is equal 125% of the flow rate into the system.



a) **(5 points)** Develop differential equations for the concentration of the reaction product in each of the two vessels. Express this as a single matrix equation.
b) **(10 points)** Solve for the concentration as a function of time in the two vessels. Do this by finding the eigenspace NOT by Laplace methods.
c) **(5 points)** If neither vessel initially has any reaction product in it, what is **c**(t)?
d) **(5 points)** What are the steady state concentrations in the vessels?

Assuming: mass of each vessel is constant (?!)

Tank 1: Acc = In − out $\Rightarrow \frac{d}{dt}(c_1 m_1) = FC_o + fFC_2 - FC$

$$\frac{dc_1}{dt} = \frac{F}{m_1}\left(-c_1 + fc_2 + C_o\right)$$

Tank 2: Acc = In − Out $\Rightarrow \frac{d}{dt}(c_2 m_2) = FC - fFC_2 - FC_2$

$$\frac{dc_2}{dt} = \frac{F}{m_2}\left(c_1 - (1+f)c_2\right)$$

$$\underline{c}' = \underline{\underline{A}}\,\underline{c} + \underline{g} \Rightarrow \underline{\underline{A}} = \begin{bmatrix} -F/m & fF/m \\ F/m_2 & -(1+f)\,F/m_2 \end{bmatrix} \qquad \underline{g} = \begin{bmatrix} FC_o/m_1 \\ 0 \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} -2 & 2.5 \\ 0.2 & -0.45 \end{bmatrix} \text{min}^{-1} \qquad \underline{g} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{kg R/kg total}$$

Problem 2 (con't)

$$\begin{vmatrix} -2-\lambda & 2.5 \\ 0.2 & -0.45-\lambda \end{vmatrix} = (-2-\lambda)(-0.45-\lambda) - 2.5(0.2)$$

$$\lambda^2 + 2.45\lambda + 0.4 = 0$$

$$\lambda = \tfrac{1}{2}\left(-2.45 \pm \sqrt{4.4025}\right)$$

$$\lambda_1 = -2.27 \qquad \lambda_2 = -0.18$$

$\lambda_1 = -2.27 \quad 0.27x_1 + 2.5x_2 = 0 \qquad \underline{x}^{(1)} = [1 \quad -0.11]^T$

$\lambda_2 = -0.18 \quad -1.82x_1 + 2.5x_2 = 0 \therefore \qquad \underline{x}^{(2)} = [\quad 0.728]^T$

Nonhomogeneous $\quad \underline{c}_p = \underline{u} \quad \underline{c}_p' = 0 \qquad \underline{0} = \underline{\underline{A}}\,\underline{u} + \underline{g}$

$$\underline{u} = \underline{\underline{A}}^{-1}(-\underline{g}) \text{ or } -\underline{\underline{A}}\,\underline{u} = \underline{g}$$

$$\left.\begin{array}{l} -2u_1 + 2.5u_2 = 2 \\ 0.2u_1 - 0.45u_2 = 0 \end{array}\right\} \rightarrow \underline{u}_1 = 2.25 \quad u_2 = 1$$

$$\underline{c}(t) = c_1\underline{x}^{(1)}e^{-2.27t} + c_2\underline{x}^{(2)}e^{-0.18t} + \begin{bmatrix} 2.25 \\ 1 \end{bmatrix}$$

At $t=0$ $\underline{c}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{array}{l} 0 = c_1 + c_2 + 2.25 \\ 0 = -0.11c_1 + 0.728c_2 + 1.0 \end{array}\right\} \rightarrow \begin{array}{l} c_1 = -0.769\ \text{kg/kg} \\ c_2 = -1.839\ \text{kg/kg} \end{array}$$

Steady state $\underline{c}' = \underline{0} \therefore \quad \underline{\underline{A}}\,\underline{c} + \underline{g} = 0 \quad \underline{c}_{ss} = \underline{\underline{A}}^{-1}(-\underline{g})$

$$\underline{c}_{ss} = \begin{bmatrix} 2.25 \\ 1 \end{bmatrix} \text{kg R/kg}$$

Of course, none of this makes physical sense! The C_o conc is meaningless, but that is not fatal. Using $C_o = 0.1$ kg R/kg total would simply shift the answer, not change its form. The bigger question is: how can m_1 & m_2 be constant?!

3. The following second order differential equation could be solved by a number of different methods. Here, we will utilize Laplace transforms to solve for the unknown function y(t).

$$y'' + a\,y' + 4\,y = 6\,e^{-2t}\;;\; y(0) = 2,\; y'(0) = 0$$

a) **(4 points)** Take the Laplace transform of the second order equation. Leave the expression for Y(s) in terms of the unknown parameter *a*.
b) **(8 points)** For the case *a* = -5, simplify and invert the function Y(s) to find y(t).
c) **(8 points)** Using the Y(s) from part (a), find y(t) for the case *a* = 0 **OR** *a* = 4. Choose only **one** of the two values of *a*.
d) **(5 points)** If the differential equation had instead been a third order linear ODE (i.e., $y^{(3)} + y'' + a\,y' + 4\,y = 6\,e^{-2t}$), what would be the general form of Y(s)? State any terms needed to determine an expression for Y(s). Do not invert.

$$s^2 Y - s y(0) - \cancel{y'(0)}^{\,0} + a\{sY - y(0)\} + 4Y = 6\,\frac{1}{s+2}$$

$$\boxed{Y(s) = \frac{2(s+a)}{s^2+as+4} + \frac{6}{(s+2)(s^2+as+4)}}$$

$a = -5$

$$s^2 - 5s + 4 \rightarrow (s-4)(s-1) \qquad Y(s) = \frac{2(s-5)(s+2)+6}{(s+2)(s-1)(s-4)}$$

$$Y(s) = \frac{A_1}{s-1} + \frac{A_2}{s-4} + \frac{A_3}{s+2} \Rightarrow A = \lim_{s\to 1} \frac{2(s-5)(s+2)+6}{(s+2)(s-4)} = \frac{2(-4)(3)+6}{3(-3)} = 2$$

$$A_2 = \lim_{s\to 4} \frac{2(s-5)(s+2)+6}{(s-1)(s+2)} = \frac{2(-1)(6)+6}{(3)(6)} = \frac{-1}{3}$$

$$A_3 = \lim_{s\to -2} \frac{2(s-5)(s+2)+6}{(s-1)(s-4)} = \frac{6}{(-3)(-6)} = \frac{1}{3}$$

$$\boxed{y(t) = 2e^{t} - \frac{1}{3}e^{4t} + \frac{1}{3}e^{-2t}}$$

c) con't

Problem 3 (con't)

$a = 4 \qquad s^2 + 4s + 4 = (s+2)(s+2) \rightarrow Y(s) = \frac{2(s+4)}{(s+2)^2} + \frac{6}{(s+2)^2}$

$Y(s) = \frac{A_1}{s+2} + \frac{A_2}{(s+2)^2} + \frac{6}{(s+2)^3} \qquad A_2 = \lim_{s \to -2} \cancel{(s+2)^2} \cdot \frac{2(s+4)}{\cancel{(s+2)^2}} = 2(2) = 4$

$A_1 = \lim_{s \to -2} (s+2)^3 \cdot \frac{d}{ds}\left[\frac{2(s+4)}{(s+2)^2}\right] =$

$\cancel{(s+2)^3}\left[\frac{2(s+4)\cdot 2\cancel{(s+2)} - 2(s+2)^{\cancel{2}}}{\cancel{(s+2)^3}}\right]$

$Y(s) = \frac{8}{s+2} + \frac{4}{(s+2)^2} + \frac{6}{(s+2)^3} \qquad A_1 = 2(2)\cdot 2 - 2(0) = 8$

$$\boxed{y(t) = 8e^{-2t} + 4te^{-2t} + 3t^2e^{-2t}}$$

$a = 0 \qquad s^2 + 4 \longrightarrow Y(s) = \frac{2s}{(s^2+4)} + \frac{6}{(s+2)(s^2+4)} = \frac{2s}{s^2+4} + \frac{3}{4}\frac{1}{s+2} - \frac{3}{4}\left(\frac{s+2}{s^2+4}\right)$

$Y(s) = \frac{\frac{1}{2}s}{s^2+4} - \frac{3}{4}\cdot\frac{2}{s^2+4} + \frac{3}{4}\cdot\frac{1}{s+2}$

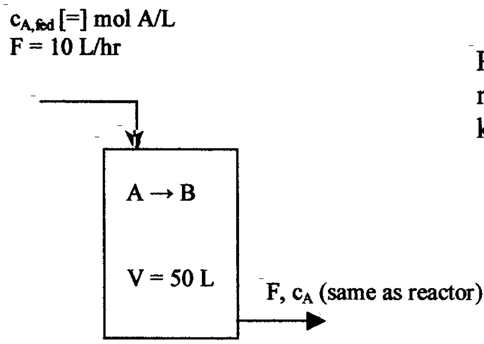
$$\boxed{y(t) = \frac{1}{2}\cos(2t) - \frac{3}{4}\sin(2t) + \frac{3}{4}e^{-2t}}$$

d) The $y^{(3)}$ term gives $s^3Y - s^2y(0) - sy'(0) - y''(0)$

↑ need this term.

$Y(s) = \frac{f(s)}{s^3 + s^2 + as + 4}$

4. A reactor is initially charged with a solution of reactant A in solvent at a concentration of $c_o = 1$ mol/L and a first order reaction started ($A \rightarrow B$). The reaction follows kinetics such that the rate of consumption is proportional to the instantaneous concentration of A. This reactor is fed with fresh reactant (A) at concentration $c_{feed}(t)$. Product is removed at the concentration in the reactor at a rate such that the overall mass in the reactor does not change.



Rate of A consumed in the reaction $A \rightarrow B$ is $k_1 c_A(t)$ where $k_1 = 0.3$ hr^{-1}.

Product streams of other reactors control the concentration of the feed such that c_{feed} changes suddenly over the course of a plant shift (total duration 8 hours). The profile of the feed concentration is as follows: there is no A in the feed stream for the first two hours of the shift, the concentration of A then jumps to 5 mol/L for the next four hours, then falls to 2 mol/L for the last 2 hours of the shift.

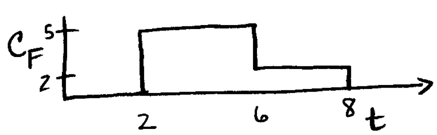
a) **(10 points)** Develop a differential equation that describes the concentration of A in the reactor as a function of time.
b) **(10 points)** Use the method of Laplace transforms to find an expression for $c_A(t)$.
c) **(5 points)** Sketch the concentration of A in the reactor as a function of time; label all pertinent points on your figure. At what point in the cycle do you expect the concentration to be highest?

Conserved property: mass of A $\quad m = c_A V \cdot M_A$ ← mol wt of A (or drop)

$$\text{Acc} = \text{In} - \text{Out} + \cancel{\text{Gen}} - \text{Cons}$$

$$V \frac{dc}{dt} = Fc_F - Fc \quad k_1 c V$$ ← units = mol/hr

$$\boxed{\frac{dc}{dt} = -\left(\frac{F}{V} + k_1\right)c + \frac{F}{V} C_F(t)}$$



$$C_F(t) = 5u(t-2) - 3u(t-6) - \underbrace{2u(t-8)}$$

ok if you did not include this term.

Problem 4 (con't)

$$c' + \left(\frac{F}{V} + k_1\right) c = \frac{F}{V}\left\{5u(t-2) - 3u(t-6) - 2u(t-8)\right\}$$

where $\frac{F}{V} + k_1 = \frac{10\ L/hr}{50\ L} + 0.3\frac{1}{hr} = 0.5\ hr^{-1}$

$\frac{F}{V} = 0.2\ hr^{-1}$ $\quad c(t=0) = c_0 = 1\ mol/L$

$$sC - c_0 + 0.5C = 5(0.2)\frac{e^{-2s}}{s} - 3(0.2)\frac{e^{-6s}}{s} - 2(0.2)\frac{e^{-8s}}{s}$$

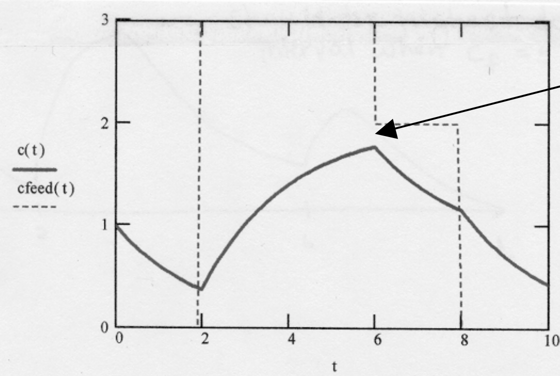
$$C = \frac{1}{(s+0.5)} + \frac{e^{-2s}}{s(s+0.5)} - 0.6\frac{e^{-6s}}{s(s+0.5)} - 0.4\frac{e^{-8s}}{s(s+0.5)}$$

each term contains $\frac{1}{s(s+0.5)} = \frac{A_1}{s} + \frac{A_2}{s+0.5}$ $\quad A_1 = \lim_{s\to 0}\frac{1}{0.5} = 2$ $\quad A_2 = \lim_{s\to -0.5}\frac{1}{-0.5} = -2$

So: $\frac{1}{s(s+0.5)} = \frac{2}{s} - \frac{2}{(s+0.5)}$

$$Y(s) = \frac{1}{(s+0.5)} + 2e^{-2s}\left\{\frac{1}{s} - \frac{1}{(s+0.5)}\right\} - 1.2e^{-6s}\left\{\frac{1}{s} - \frac{1}{(s+0.5)}\right\} - 0.8e^{-8s}\left\{\frac{1}{s} + \frac{1}{(s+0.5)}\right\}$$

$$\boxed{y(t) = e^{-0.5t} + 2u(t-2)\left\{1 - e^{-0.5(t-2)}\right\} - 1.2u(t-6)\left\{1 - e^{-0.5(t-6)}\right\} - 0.8u(t-8)\left\{1 - e^{-0.5(t-8)}\right\}}$$



Concentration reaches a maximum at some point after the inlet concentration jumps to 5 mol/L. The concentration will increase until as the reaction rate is now slower than the inlet flow.