1. The following third order differential equation describes the inertial response of the cantilever in an atomic force microscope:

\[ y^{(3)} + 2y'' + 9y' + 18y = 18(2t - 1); \ y(0) = 0, y'(0) = 1, y''(0) = 1 \]

a) (5 points) Convert this third order equation in y(x) to a system of coupled first order differential equations. Identify all elements in the matrices and vectors.

b) (8 points) The eigenvalues for this coupled system are -2, 3i, and -3i. Find the general solution to the homogeneous coupled system.

c) (7 points) Solve for the solution of the nonhomogeneous, coupled system using the method of Undetermined Coefficients.

d) (5 points) Use the initial conditions to solve for the arbitrary constants.

\[ \begin{cases}
  y_1 = y & \frac{dy_1}{dt} = y_2 \\
  y_2 = y' & \frac{dy_2}{dt} = y_3 \\
  y_3 = y'' & \frac{dy_3}{dt} = -2y_3 - 9y_2 - 18y + 18(2t - 1)
\end{cases} \]

Find \( x \)'s for \( \lambda_1 = -2, \lambda_2 = 3i, \lambda_3 = -3i \):

\( \lambda_1 = -2 \)

\[ \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -18 & -9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \]

\[ 2x_1 = -x_2 \]

\[ 2x_2 = -x_3 \]

\[ x = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \]

\( \lambda_2 = 3i \)

\[ \begin{bmatrix} 3i & 1 & 0 \\ 0 & 3i & 1 \\ -18 & -9 & -2 - 3i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \]

\[ -3i x_1 = -x_2 \]

\[ -3i x_2 = -x_3 \]

\[ x = \begin{bmatrix} 1 \\ 3i \\ -9 \end{bmatrix} \]

\( \lambda_3 = -3i \)

\[ \begin{bmatrix} -3i & 0 \\ 0 & 3i & 1 \\ -18 & -9 & -2 + 3i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \]

\[ 3i x_1 = -x_2 \]

\[ 3i x_2 = -x_3 \]

\[ x = \begin{bmatrix} 1 \\ -3i \\ -9 \end{bmatrix} \]

\[ y(t) = c_1 x_1 e^{-2t} + c_2 x_2 e^{3i t} + c_3 x_3 e^{-3i t} \]

\[ y(t) = c_1 x_1 e^{-2t} + c_2 x_2 e^{3i t} + c_3 x_3 e^{-3i t} \]

** Matrices defined below

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & -9 & -2 \end{bmatrix} \] and \[ g = 18 \begin{bmatrix} 0 \\ 0 \\ 2t - 1 \end{bmatrix} \] and \[ y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \]
Problem 1 (con’t)

\[
\begin{bmatrix}
-1 \\
3i \\
9
\end{bmatrix} e^{3it} = \begin{bmatrix}
-1 \\
3i \\
9
\end{bmatrix} \left[ \cos(3t) + i \sin(3t) \right] = \begin{bmatrix}
-\cos(3t) \\
3 \sin(3t) \\
9 \cos(3t)
\end{bmatrix} + \begin{bmatrix}
-\sin(3t) \\
3 \cos(3t) \\
9 \sin(3t)
\end{bmatrix}
\]

\[y_h(x) = C_1 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} -\cos(3t) \\ -3 \sin(3t) \\ 9 \cos(3t) \end{bmatrix} + C_3 \begin{bmatrix} -\sin(3t) \\ 3 \cos(3t) \\ 9 \sin(3t) \end{bmatrix}\]

Nonhomogeneous: assume \(y_p = u + vt\) \(\Rightarrow\) \(y'_{dp} = v\)

\[v = A(u + vt) + g\] where \(A = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, g = \begin{bmatrix} 0 \\ 0 \\ -18 \end{bmatrix}\)

\(\Rightarrow t \text{ terms: } -A \cdot v = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} v_2 = 0 \\ v_3 = 0 \\ -3v_1 - 9v_2 - 2v_3 = 36 \end{cases} \Rightarrow v_1 = 2\)

\(\Rightarrow t^0 \text{ terms: } v - A \cdot u = \begin{bmatrix} 0 \\ 0 \\ -18 \end{bmatrix} \Rightarrow \begin{cases} 2 - u_2 = 0 \\ 0 - u_3 = 0 \\ 18u_1 + 9u_2 + 2u_3 = -18 \end{cases} \Rightarrow u_2 = 2, u_3 = 0, u_1 = -2\)

\[y_{dp} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} t\]

\[y = y_h + y_{dp} \quad \Rightarrow \quad y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\]

\[y(0) = C_1 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \\ 9 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\]

\[c_1 - c_2 - 2 = 0 \\
-2c_1 + 3c_3 + 2 = 1 \\
4c_1 + 9c_2 = 1\]

\[\Rightarrow \quad c_1 = \frac{19}{13}, \quad c_2 = \frac{-7}{13}, \quad c_3 = \frac{25}{13}\]
2. Reaction products are often concentrated in a stage-wise manner to avoid the dangers of product overheating caused by large exothermic heats of dilution. A hot reaction product is pumped into a sequence of two vessels (also known as tanks). The purpose of the system is to concentrate the reaction product $R$ solution. The first vessel is 10 times smaller than the second and reaction product is recycled within the two vessels. The recycle rate is equal 125% of the flow rate into the system.

![Diagram of the system](image)

$F = 10 \text{ kg/min}$  
$C_0 = 1 \text{ kg R/kg total}$  
$m_1 = 5 \text{ kg}$  
$m_2 = 50 \text{ kg}$  
$C_1(t)$

\[ \frac{dC_1}{dt} = \frac{F}{m_1} \left( -C_1 + fC_2 + C_0 \right) \]

\[ \frac{dC_2}{dt} = \frac{F}{m_2} \left( C_1 - (1+f)C_2 \right) \]

\[ c' = A \cdot c + g \]

\[ A = \begin{bmatrix} -2 & 2.5 \\ 0.2 & -0.45 \end{bmatrix} \text{ min}^{-1} \]

\[ g = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ kg R/kg total} \]

\[ A = 1.25 \]

\[ \frac{dC}{dt} = FC_0 + \int FC \]

\[ C(t) \]

\[ F = 10 \text{ kg/min} \]

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a) (5 points) Develop differential equations for the concentration of the reaction product in each of the two vessels. Express this as a single matrix equation.

b) (10 points) Solve for the concentration as a function of time in the two vessels. Do this by finding the eigenspace NOT by Laplace methods.

c) (5 points) If neither vessel initially has any reaction product in it, what is $c(t)$?

d) (5 points) What are the steady state concentrations in the vessels?
Problem 2 (con’t) \[ \begin{align*}
-2 - \lambda & = 2.5 \\
0.2 & = -0.45 - \lambda \\
\lambda^2 + 2.45 \lambda + 0.4 & = 0 \\
\lambda & = \frac{1}{2} (-2.45 \pm \sqrt{4.4025}) \\
\lambda_1 & = -2.27 \\
\lambda_2 & = -0.18
\end{align*} \]

\[ \lambda_1 = -2.27 \quad \lambda_2 = -0.18 \]

\[ x^{(0)} = \begin{bmatrix} 1 \\ -0.11 \end{bmatrix}^T \]

\[ x^{(2)} = \begin{bmatrix} 0.728 \end{bmatrix}^T \]

Nonhomogeneous \[ \begin{align*}
C_p & = U \\
C'_p & = 0 \\
o & = A u + q \\
u & = A^{-1} (-q) \text{ or } -Au = q
\end{align*} \]

\[-2u_1 + 2.5u_2 = 2 \\
0.2u_1 - 0.45u_2 = 0 \]

\[ \begin{align*}
u_1 & = 2.25 \\
u_2 & = 1
\end{align*} \]

\[ C(t) = c_1 x^{(0)} e^{-2.27t} + c_2 x^{(2)} e^{-0.11t} + \begin{bmatrix} 2.25 \\ 1 \end{bmatrix} \]

At \[ t=0 \] \[ C(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \begin{align*}
o & = c_1 + c_2 + 2.25 \\
o & = -0.11c_1 + 0.728c_2 + 1.0 \end{align*} \]

\[ \begin{align*}
c_1 & = -0.769 \text{ kg} \\
c_2 & = -1.839 \text{ kg}
\end{align*} \]

Steady state \[ C' = 0 \] \[ A C + q = 0 \]

\[ C_{ss} = A^{-1} (-q) \]

\[ C_{ss} = \begin{bmatrix} 2.25 \\ 1 \end{bmatrix} \text{ kg R/kg} \]

Of course, none of this makes physical sense! The \( C_0 \) conc is meaningless, but that is not fatal. Using \( C_0 = 0.1 \text{ kg R/kg} \) total would simply shift the answer, not change its form. The bigger question is: how can \( m_1 \) and \( m_2 \) be constant?!
3. The following second order differential equation could be solved by a number of different methods. Here, we will utilize Laplace transforms to solve for the unknown function \( y(t) \).

\[
\frac{d^2y}{dt^2} + ay' + 4y = 6e^{-2t}; \quad y(0) = 2, \quad y'(0) = 0
\]

\( a \) is a constant.

a) **(4 points)** Take the Laplace transform of the second order equation. Leave the expression for \( Y(s) \) in terms of the unknown parameter \( a \).

b) **(8 points)** For the case \( a = -5 \), simplify and invert the function \( Y(s) \) to find \( y(t) \).

c) **(8 points)** Using the \( Y(s) \) from part (a), find \( y(t) \) for the case \( a = 0 \) or \( a = 4 \). Choose only one of the two values of \( a \).

d) **(5 points)** If the differential equation had instead been a third order linear ODE (i.e., \( y''' + y'' + ay' + 4y = 6e^{-2t} \)), what would be the general form of \( Y(s) \)? State any terms needed to determine an expression for \( Y(s) \). Do not invert.

\[
Y(s) = \frac{2(s + a)}{s^2 + 4s + 4} + \frac{6}{(s+2)(s^2 + 4s + 4)}
\]

\( a = -5 \)

\[
s^2 - 5s + 4 \rightarrow (s-4)(s-1)
\]

\[
Y(s) = \frac{2(s-5)(s+2) + 6(s+2)(s-1)}{(s-4)(s-1)(s-5)}
\]

\[
A_1 = \lim_{s \to 1} \frac{2(s-5)(s+2) + 6}{(s-4)(s-1)} = \frac{2(-4)(3) + 6}{3(-3)} = \frac{2}{3}
\]

\[
A_2 = \lim_{s \to 4} \frac{2(s-5)(s+2) + 6}{(s-1)(s+2)} = \frac{2(-1)(6) + 6}{3(6)} = \frac{1}{3}
\]

\[
A_3 = \lim_{s \to -2} \frac{2(s-5)(s+2) + 6}{(s-1)(s-5)} = \frac{6}{(-3)(-6)} = \frac{1}{3}
\]

\[
y(t) = 2e^{-\frac{1}{3}t} - \frac{4}{3}e^{-2t} + \frac{1}{3}e^{-2t}
\]
c) \( a = 4 \) \( s^2 + 4s + 4 = (s + 2)(s + 2) \) \( \Rightarrow \) \( Y(s) = \frac{2(s + 4)}{(s + 2)^2} + \frac{6}{(s + 2)^3} \)

\[ \begin{align*}
A_2 &= \lim_{s \to 2} \frac{2(s + 4)}{(s + 2)^2} = 2(2) = 4 \\
A_1 &= \lim_{s \to -2} \frac{2(s + 4)}{(s + 2)^2} = -2(2) = -4
\end{align*} \]

\[ y(t) = 8e^{-2t} + 4te^{-2t} + 3e^{-2t} \]

\[ a = 0 \quad s^2 + 4 \quad \Rightarrow \quad Y(s) = \frac{2s}{(s^2 + 4)} + \frac{6}{(s + 2)(s^2 + 4)} = \frac{2s}{s^2 + 4} + \frac{3}{4} \quad \frac{1}{s + 2} - \frac{3}{4} \quad \frac{s + 2}{s^2 + 4} \]

\[ y(t) = \frac{1}{2} \cos(2t) - \frac{3}{4} \sin(2t) + \frac{3}{4} e^{-2t} \]

d) The \( y^{(3)} \) term gives \( s^3 Y - s^2 y(0) - sy'(0) - y''(0) \) \( \uparrow \) need this term

\[ Y(s) = \frac{f(s)}{s^3 + s^2 + 4s + 4} \]
4. A reactor is initially charged with a solution of reactant A in solvent at a concentration of $c_A = 1 \text{ mol/L}$ and a first order reaction started ($A \rightarrow B$). The reaction follows kinetics such that the rate of consumption is proportional to the instantaneous concentration of A. This reactor is fed with fresh reactant (A) at concentration $c_{\text{feed}}(t)$. Product is removed at the concentration in the reactor at a rate such that the overall mass in the reactor does not change.

\[
\begin{align*}
&c_{\text{feed}} = \text{mol A/L} \\
&F = 10 \text{ L/hr}
\end{align*}
\]

Rate of A consumed in the reaction $A \rightarrow B$ is $k_1 c_A(t)$ where $k_1 = 0.3 \text{ hr}^{-1}$.

Product streams of other reactors control the concentration of the feed such that $c_{\text{feed}}$ changes suddenly over the course of a plant shift (total duration 8 hours). The profile of the feed concentration is as follows: there is no A in the feed stream for the first two hours of the shift, the concentration of A then jumps to 5 mol/L for the next four hours, then falls to 2 mol/L for the last 2 hours of the shift.

a) (10 points) Develop a differential equation that describes the concentration of A in the reactor as a function of time.

b) (10 points) Use the method of Laplace transforms to find an expression for $c_A(t)$.

c) (5 points) Sketch the concentration of A in the reactor as a function of time; label all pertinent points on your figure. At what point in the cycle do you expect the concentration to be highest?

Conserved property: mass of A

\[
m = c_A V N_A \text{ mol wt of A}
\]

\[
\text{Acc} = \text{In} - \text{Out} + \text{Gen} - \text{Cons}
\]

\[
V \frac{dc}{dt} = FC_F - FC - k_1 c V \quad \text{units} = \text{mol/hr}
\]

\[
\frac{dc}{dt} = -\left(\frac{F}{V} + k_1\right)c + \frac{F}{V} C_F(t)
\]

\[
C_F(t) = 5u(t-2) - 3u(t-6) - 2u(t-8)
\]

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Concentration reaches a maximum at some point after the inlet concentration jumps to 5 mol/L. The concentration will increase until as the reaction rate is now slower than the inlet flow.