Proof Theory and Proof Mining II: Formal Theories of Analysis

Jeremy Avigad

Department of Philosophy and
Department of Mathematical Sciences
Carnegie Mellon University

July 2015

Sequence of Topics

1. Computable Analysis
2. Formal Theories of Analysis
3. The Dialectica Interpretation and Applications
4. Ultraproducts and Nonstandard Analysis

Formal theories of analysis

Analysis can be formalized in set theory.

But Part I showed that we don't always need big sets: many objects of analysis can be represented by sets of natural numbers.

Hilbert and Bernays' *Grundlagen der Mathematik*, volume II, 1936, showed that portions of analysis can be formalized in second-order arithmetic (with third-order parameters).

SOA is often called "analysis" for that reason.

Kreisel: provability in restricted theories provides additional information.

Axiomatic theories

We will consider:

- Primitive recursive arithmetic (PRA)
- First-order arithmetic (PA, HA) (and subsystems)
- Second-order arithmetic (PA^2, HA^2) (and subsystems)

These come in classical and intuitionistic versions.

We will also consider theories of finite type.

Primitive recursive arithmetic

Recall that the primitive recursive functions include

- zero, 0
- successor, S
- projections, $p_i^n(x_1, \ldots, x_n) = x_i$

and are closed under

- composition: $f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_n(\vec{x}))$
- primitive recursion:

$$f(0, \vec{z}) = g(\vec{z}), \quad f(x+1, \vec{z}) = h(f(x, \vec{z}), x, \vec{z})$$

Primitive recursive arithmetic

Primitive recursive arithmetic is an axiomatic theory, with

- $0 \neq S(x)$, $S(x) = S(y) \to x = y$
- defining equations for the primitive recursive functions
- quantifier-free induction:

$$\frac{\varphi(0) \qquad \varphi(x) \to \varphi(x+1)}{\varphi(t)}$$

In *PRA*, can handle "finitary" proofs.

Primitive recursive arithmetic

Just as all "reasonable" computable functions are primitive recursive, all "reasonable" facts about them can be proved in *PRA*.

In fact, it is surprisingly hard to find ordinary mathematical theorems that can be stated in the language of *PRA* but not proved there.

Primitive recursive arithmetic

PRA can be presented either as a first-order theory (classical or intuitionistic) or as a quantifier-free calculus.

Theorem. Suppose first-order classical *PRA* proves $\forall x \, \exists y \; \varphi(x, y)$, with φ quantifier-free. Then for some function symbol f, quantifier-free *PRA* proves $\varphi(x, f(x))$.

Proof. First, show *PRA* has a set of universal axioms. Define bounded quantification, and replace induction by

$$\forall y \; (\varphi(0) \wedge \forall x < y \; (\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \varphi(y)).$$

Then apply Herbrand's theorem.

Herbrand's theorem

Theorem. Suppose T is a universal axiomatized first-order theory, and $T \vdash \exists \vec{x}\, \varphi(\vec{x})$, φ quantifier free. Then there are sequences of terms $\vec{t_1}, \ldots \vec{t_n}$ and a quantifier-free proof of

$$\varphi(\vec{t_1}) \vee \ldots \vee \varphi(\vec{t_n})$$

from instances of axioms of T.

Proof. WLOG, assume T has a constant symbol.

If the conclusion fails, $T \cup \{\neg\varphi(\vec{t})\}$ is consistent, where $\vec{t}$ ranges over all tuples of terms.

Let $\mathcal{M}$ be a model. Let $\mathcal{M}'$ be the submodel with universe $\{t^{\mathcal{M}}\}$. Then $\mathcal{M}' \models T \cup \{\forall \vec{x}\, \neg\varphi(\vec{x})\}$.

First-order arithmetic

First-order arithmetic is essentially *PRA* plus induction. *Peano arithmetic* (*PA*) is classical, *Heyting arithmetic* (*HA*) is intuitionistic.

Language: $0, S, +, \times, <$.

Axioms: quantifier-free defining axioms, induction.

A formula is

- Δ_0 if every quantifier is bounded
- Σ_1 if of the form $\exists \vec{x}\, \varphi$, $\varphi \in \Delta_0$
- Π_1 if of the form $\forall \vec{x}\, \varphi$, $\varphi \in \Delta_0$
- Δ_1 if equivalent to Σ_1 and Π_1

Primitive recursive functions / relations have Σ_1/Δ_1 definitions.

First-order arithmetic

$I\Sigma_1$ is the restriction of PA with induction for only Σ_1 formulas.

This theory suffices to define the primitive recursive functions, and hence interpret PRA. Conversely:

Theorem (Parsons, Mints, Takeuti). $I\Sigma_1$ is conservative over PRA for Π_2 sentences: if

$$I\Sigma_1 \vdash \forall x \, \exists y \, \varphi(x, y),$$

with φ quantifier-free, then

$$PRA \vdash \varphi(x, f(x))$$

for some function symbol f.

Conservativity of $I\Sigma_1$ over PRA

There are various ways to prove this theorem.

Syntactic proofs:

- Using cut elimination or normalization.
- Using the Dialectica interpretation (plus normalization).

Model-theoretic proofs:

- A model-theoretic argument due to Friedman.
- A "saturation" construction.

Double-negation translations

The Gödel-Gentzen double-negation translation interprets classical logic in minimal logic:

- $A^N \equiv \neg\neg A$ for atomic A
- $(\varphi \vee \psi)^N \equiv \neg(\neg\varphi^N \wedge \neg\psi^N)$
- $(\exists x\ \varphi)^N \equiv \neg\forall x\ \neg\varphi^N$.

The translation commutes with $\forall$, $\wedge$, $\rightarrow$.

Theorem. If $\Gamma \vdash \varphi$ classically, $\Gamma^N \vdash \varphi^N$ in minimal logic.

The proof uses induction on derivations. For example, $(\varphi \vee \neg\varphi)^N$ is easily proved.

Reducing PA to HA

Theorem. If PA proves a formula, φ, then HA proves φ^N.

Proof. For each axiom of φ of PA, HA proves φ^N.

Corollary. If PA proves $\forall x \, \exists y \; R(x, y)$, where R is primitive recursive, then HA proves $\forall x \, \neg \forall y \, \neg R(x, y)$.

Better theorem. If PA proves $\forall x \, \exists y \; R(x, y)$, where R is primitive recursive, then HA proves $\forall x \, \exists y \; R(x, y)$.

There are various ways to prove this; later, we will use the Dialectica interpretation.

Second order arithmetic

The language is two-sorted:

- variables $x, y, z, \ldots$ and functions $0, S, +, \times$ on one sort
- variables $X, Y, Z, \ldots$ on the other sort
- a relation $t \in X$ between the two sorts

Axioms:

- axioms of PA, with induction extended to the bigger language
- comprehension: $\exists X \, \forall y \, (y \in X \leftrightarrow \varphi(y, \vec{z}))$

The "standard model" is $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \ldots)$, but there are smaller ones.

An ω-model is a model where the first-order part is standard, i.e. $\mathbb{N}$.

Second order arithmetic

We can *define* equality on the second sort by

$$X = Y := \forall z \, (z \in X \leftrightarrow z \in Y).$$

The usual axioms of equality (including substitution) follow.

This is known as "extensional equality."

Alternatively, we can take equality as a basic symbol, and add the axiom above.

This second system is interpreted in the first.

Second order arithmetic

Functions vs. sets:

- With functions basic, interpret sets as characteristic functions:

$$x \in S \equiv \chi_S(x) = 1.$$

- With relations basic, interpret functions as functional relations, $\forall x \; \exists! y \; R(x, y)$

Can add choice axioms:

$$\forall x \; \exists y \; \varphi(x, y) \to \exists f \; \forall y \; \varphi(x, f(x)).$$

Second order arithmetic

From a proof-theoretic perspective, second-order arithmetic is very strong.

We obtain weaker systems by:

- restricting comprehension
- restricting induction

Subsystems of second-order arithmetic

The big five:

- RCA_0: recursive (Δ^0_1) comprehension
  (formalized computable analysis)
- WKL_0: weak König's lemma
  (a form of compactness)
- ACA_0: arithmetic comprehension
  (analytic principles like the least-upper bound principle.)
- ATR_0: transfinitely iterated arithmetic comprehension
  (transfinite constructions)
- $\Pi^1_1-CA_0$: Π^1_1 comprehension
  (strong analytic principles)

We will focus on the first three.

RCA_0

The axioms of RCA_0 are as follows:

- quantifier-free axioms for $0, S, +, \times, <$
- induction, restricted to Σ_1 formulas (with both number and set parameters):

$$\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x\, \varphi(x)$$

- the recursive comprehension axiom, (RCA):

$$\forall x\, (\varphi(x) \leftrightarrow \psi(x)) \rightarrow \exists Y\, \forall x\, (x \in Y \leftrightarrow \varphi(x))$$

where φ is Σ_1 and ψ is Π_1.

RCA_0

Notice that the induction schema includes set induction:

$$0 \in Y \wedge \forall x\,(x \in Y \to x+1 \in Y) \to \forall x\,(x \in Y).$$

It is slightly stronger.

Since RCA_0 includes $I\Sigma_1$, we can act as though primitive recursive arithmetic is "built-in."

RCA_0

(*RCA*) says that if a set exists if it has a computably enumerable definition as well as a co-computably enumerable definition (relative to others sets in the universe).

Roughly, it allows you to define computable sets and relations.

Let *REC* denote the set of recursive sets. Then $(\mathbb{N}, \mathit{REC}, \ldots)$ is the minimal ω-model.

Analysis in *RCA_0* is roughly “formalized computable analysis.”

WKL_0

Remember that a *tree on* $\{0,1\}$ is a set T of finite binary sequences closed under initial segments:

$$Tree(T) \equiv \forall \sigma, \tau \; (\sigma \in T \wedge \tau \subseteq \sigma \rightarrow \tau \in T).$$

We can say T is infinite as follows:

$$Infinite(T) \equiv \forall n \; \exists \sigma \; (\sigma \in T \wedge length(\sigma) = n).$$

A set P is a *path through* T if, when viewed as an infinite binary sequence, every initial segments is in T:

$$Path(P, T) \equiv \forall \sigma \; ((\forall i < length(\sigma) \; , i \in P \leftrightarrow (\sigma)_i = 1) \rightarrow \sigma \in T).$$

WKL_0

Weak König's lemma (WKL) says that every infinite binary tree has a path:

$$\forall T\ (Tree(T) \wedge Infinite(T) \rightarrow \exists P\ Path(P, T)).$$

The theory WKL_0 is $RCA_0 + (WKL)$.

WKL_0

Since there are computable infinite binary trees with no computable path, *REC* is *not* an ω-model.

It has a model where every set is computable from $0'$.

Remarkably, (*WKL*) has no minimal ω-model, but the intersection of all ω-models is *REC*.

WKL_0

Over RCA_0, (WKL) is equivalent to each of these:

- the Heine-Borel theorem (for $[0, 1]$)
- Every open cover of $\{0, 1\}^\omega$ has a finite subcover.
- Every continuous function on $[0, 1]$ is uniformly continuous
- Every continuous function on $[0, 1]$ is bounded

Here, $[0, 1]$ can be replaced by any compact space.

(More on this below.)

ACA_0

A formula is *arithmetic* if it has only first-order quantifiers.

ACA_0 adds to RCA_0 the *arithmetic comprehsion axiom*, (ACA):

$$\exists Y\,\forall x\,(x \in Y \leftrightarrow \varphi(x))$$

where φ is arithmetic (possibly with number and set parameters).

The smallest ω-model is the collection of arithmetically definable sets.

Notice that set-induction now gives us induction for every arithmetic formula.

ACA_0

Theorem. Over RCA_0, (ACA) is equivalent to each of these:

- Every bounded increasing sequence of real numbers has a least upper bound.
- Every bounded sequence of real nubmers has a least upper bound.
- Every bounded sequence of real numbers has a convergence subsequence.
- Every sequence of points in a compact metric space has a convergent subsequence.

Metamathematics of SOSOA

We have considered subsystems of second-order arithmetic from the point of view of:

- their minimal models
- what they can be prove

Knowing that a mathematical theorem is provable in a restricted theory, T, provides additional information.

Conservation results

Many central proof theoretic results have the following form:

For any $\varphi \in \Gamma$, *if* $T_1 \vdash \varphi$, *then* $T_2 \vdash \varphi'$.

These can be used to reduce

- infinitary theories to finitary ones
- classical theories to constructive ones
- impredicative theories to predicative ones
- nonstandard theories to standard ones
- higher-order theories to first-order ones
- first-order theories to quantifier-free ones

Conservation results

From a *foundational* point of view, conservations results *explain* or *interpret* theories that are more

- abstract,
- mysterious, or
- dubious,

in terms of ones that are more

- concrete,
- familiar, or
- trustworthy.

From a *practical* point of view, they provide additional information.

Conservation results

We have already seen a few examples:

- *PRA* is conservative over quantifier-free *PRA* for Π_2 formulas (in an appropriate sense)
- $I\Sigma_1$ is conservative over *PRA* for Π_2 formulas.
- If *PA* proves φ, *HA* proves φ^N.

Conservation results

There are at least two approaches to proving “if T_1 proves φ, then T_2 proves φ'”:

- *Syntactic (proof theoretic)*: translate a proof of φ in T_1 to a proof of φ in T_2.
- *Semantic (model theoretic)*: transform a model of $T_2 \cup \{\neg\varphi\}$ into a model of $T_1 \cup \{\neg\varphi\}$.

The second relies on soundness and completeness, and may *a priori* provide no information about how to translate proofs.

A slight variant works for intuitionistic theories.

Conservation results

Syntactic approaches:

- “Global” transformations
- “Local” interpretations

Global transformations like cut-elimination and normalization allow iterated exponential increase in proof length.

Local interpretations proceed line by line.

Conservation results

Interpretations:

- Double-negation translations
- The Friedman-Dragalin interpretation
- Semantic interpretations
- Forcing translations
- Realizability interpretations
- Functional interpretations

The last two work best on intuitionistic theories.

Metamathematics of RCA_0

RCA_0 is interpretable in $I\Sigma_1$.

Simply interpret the set variables as ranging over codes for computable sets.

For example, we can take the codes to be indices of Turing machines that halt on every input and return 0 or 1.

So RCA_0 is conservative over $I\Sigma_1$ for all first-order sentences.

Hence it is conservative over PRA for Π_2 sentences.

So whenever RCA_0 proves $\forall x\ \exists y\ R(x, y)$, R primitive recursive, there is a primitive recursive function witnessing this.

Weak König's lemma

Theorem (Friedman). WKL_0 is conservative over PRA for Π_2 sentences.

First syntactic proof, using cut-elimination, by Sieg.

We'll later see a proof by Kohlenbach using the Dialectica interpretation.

Theorem (Harrington). WKL_0 is conservative over RCA_0 for Π^1_1 sentences.

Harrington's model-theoretic proof is inspired by the low-basis theorem.

Syntactic proofs by Hájek, Avigad, Ferreira and Ferreira.

Metamathematics of ACA_0

Theorem. ACA_0 is conservative over PA.

Proof. Suppose PA doesn't prove φ. Let $\mathcal{M}$ be a model of $PA \cup \{\neg\varphi\}$.

Turn this into a model $ACA_0 \cup \{\neg\varphi\}$, taking the second-order part to be the collection of sets that are arithmetically definable from parameters.

It is not hard to see that (ACA) and the broader schema of induction hold in this model, while the truth value of φ does not change.

(A slight variation shows conservativity for Π^1_1 sentences, in an appropriate sense.)

Metamathematics of ACA_0

We will see later that whenever PA proves $\forall x \; \exists y \; R(x, y)$ for a primitive recursive R, then there is a primitive recursive *functional* f of type $\mathrm{N} \to \mathrm{N}$ such that $\forall x \; R(x, f(x))$ holds.

As a corollary, the same holds for ACA_0.

In other words, the *probably total computable functions* of

- RCA_0 and WKL_0 are primitive recursive, and those of
- ACA_0 can be defined by primitive recursion in the higher types.

Formalizing analysis

In the language of *PRA*, one can define integers, rational numbers, and other finitary objects in natural ways.

Define the real numbers to be Cauchy sequences of rationals with a fixed rate of convergence:

$$\forall n \, \forall m \geq n \, (|a_n - a_m| < 2^{-n}).$$

Equality is a Π_1 notion:

$$a = b \equiv \forall n \, (|a_n - b_n| \leq 2^{-n+1}).$$

Less-than is a Σ_1 notion:

$$a < b \equiv \exists n \, (a_n + 2^{-n+1} < b_n).$$

Complete separable metric spaces

Definition. A *complete separable metric space* $X = \widehat{A}$ consists of a set A together with a function $d : A \times A \to \mathbb{R}$ satisfying:

- $d(x, x) = 0$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$.

A *point of* $\widehat{A}$ is a sequence (a_n) of elements of A such that for every n and $m \geq n$ we have $d(a_n, a_m) < 2^{-n}$.

Outside the theory, we can think of A as "coding" or "representing" the metric space.

Within the theory, we say that A "is" the space.

Compactness

Three notions of compactness for a CSM:

- Totally bounded: for every rational $\varepsilon > 0$, there is a finite ε-net.
- Heine-Borel compact: every covering by open sets has a finite subcover.
- Sequentially compact: every sequence has a convergent subsequence.

In weak theories:

- RCA_0 proves e.g. $[0, 1]$ is totally bounded.
- Totally bounded $\Rightarrow$ Heine-Borel requires weak König's lemma.
- Totally bounded $\Rightarrow$ sequentially compact requires arithmetic comprehension.

In constructive mathematics, one usually uses "totally bounded."

Continuity

A function f between CSM's is *uniformly continuous* if

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x, y \, (d(x, y) < \delta \to d(f(x), f(y)) < \varepsilon).$$

A *modulus of uniform continuity* for f is a function $g(\varepsilon)$ returning such a δ for each ε:

$$\forall x, y, \varepsilon > 0 \, (d(x, y) < g(\varepsilon) \to d(f(x), f(y)) < \varepsilon).$$

Theorem. In RCA_0, the statement that every continuous function from a compact space to $\mathbb{R}$ has modulus of uniform continuity is equivalent to (WKL).

In Bishop's constructive mathematics, functions are assumed to come with such moduli ("avoidance of pseudo-generality").

Closed sets

Two notions of a closed set:

- *closed* = complement of a sequence of basic open balls
- *separably closed* = the closure of a sequence of points

Theorem (Brown). Over RCA_0:

1. For compact spaces, "closed $\Rightarrow$ separably closed" is equivalent to (ACA).
2. In general, "closed $\Rightarrow$ separably closed" is equivalent to $(\Pi^1_1 - CA)$.
3. "separably closed $\Rightarrow$ closed" is equivalent to (ACA)

Distance

Theorem (Avigad and Simic): Over RCA_0, the following are equivalent to (ACA):

1. In a compact space, if C is any closed set and x is any point, then $d(x, C)$ exists.
2. If C is any closed subset of $[0, 1]$, then $d(0, C)$ exists.

The following are equivalent to $(\Pi^1_1 - CA)$:

1. In an arbitrary space, if C is any closed set and x is any point, then $d(x, C)$ exists.
2. In a compact space, if S is any G_δ set and x is any point, then $d(x, S)$ exists.
3. If S is a G_δ subset of $[0, 1]$, then $d(0, S)$ exists.

In constructive mathematics, sets are often assumed to be *located*.

Hilbert space and Banach spaces

As in computable analysis, we can define a *Hilbert space* $H = \widehat{A}$ to be a countable vector space A over $\mathbb{Q}$ together with a function $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{R}$ satisfying

1. $\langle x, x \rangle \geq 0$
2. $\langle x, y \rangle = \langle y, x \rangle$
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

Define $||x|| = \langle x, x \rangle^{1/2}$ and $d(x, y) = ||x - y||$, and think of H as the completion of A.

Reverse mathematics of ergodic theory

Recall the mean ergodic theorem: let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive map on a Hilbert space, and for each n, let

$$A_n f = \frac{1}{n}(f + Tf + \ldots + T^{n-1}f).$$

The mean ergodic theorem asserts that this sequence of averages converges in the Hilbert space norm.

Over RCA_0, this is equivalent to (ACA).

Reverse mathematics of ergodic theory

More precisely:

- Let $M = \{f \in \mathcal{H} \mid Tf = f\}$
- Let N be the closure of $\{Tg - g \mid g \in \mathcal{H}\}$

The mean ergodic theorem says:

- M is the orthogonal complement of N
- $A_n f$ converges in norm to $P_M f$

Reverse mathematics of ergodic theory

Consider the three statements:

1. $A_n f$ converges
2. $P_N f$ exists
3. $P_M f$ exists

Theorem (Avigad and Simic) In RCA_0, 1 and 2 are equivalent, but showing that 3 implies either 1 or 2 requires (ACA).

In fact, even the statement "if $P_M f = 0$, then $A_n f$ converges" requires (ACA).

Higher types

Recall that the *finite types* are defined as follows:

- N is a finite type
- If σ and τ are finite types, so are $\sigma \times \tau$ and $\sigma \to \tau$

The *primitive recursive functionals of finite type* allow:

- λ abstraction, application, pairing, projection
- Higher-type primitive recursion:

$$F(0) = G, \quad F(n+1) = H(F(n), n)$$

Higher-type arithmetic

The theory PRA^{ω} (a.k.a. Gödel's theory T) axiomatizes these, just as PRA axiomatizes the primitive recursive functions.

- There is a sort for each type.
- Basic constants and combinators allow us to define functions using λ abstraction, application, pairing, projection.
- "Recursors" allow definition by primitive recursion.

Higher-type arithmetic

Higher-type arithmetic:

- $PA^\omega = PRA^\omega +$ induction
- $HA^\omega = PRA_i^\omega +$ induction

These are conservative extensions of PA and HA respectively.

In fact, one can add quantifier-free choice axioms $(QF{-}AC)$ to PA^ω, and full choice (AC) to HA^ω.

Restricted versions are conservative over PRA:

- $\widehat{PRA}^\omega + (QF{-}AC)$
- $\widehat{PRA}_i^\omega + (AC)$

We will obtain even stronger results with the Dialectica interpretation.

Extensionality

One can take type N equality to be basic, then define higher-type equality extensionally:

$$f = g \equiv \forall x\,(fx = gx)$$

The extensionality axiom says that functions respect this equality:

$$f = g \to Ff = Fg.$$

Alternatively, one can take $=$ to be basic at each type, and have axioms asserting that this corresponds to the extensional notion.

Theorem (Luckhardt). Extensionality can be interpreted, preserving the second-order (type $0/1$) fragment.

More generally, though, the issues are subtle.

Second-order vs. finite types

Notice that adding higher types alone doesn't make a theory strong.

It is the presence of *set comprehension* that makes second-order arithmetic so much stronger than arithmetic.

Indeed, PRA^ω can be interpreted in HA, by internalizing either HEO or HRO.

Conservation results summarized

The following theories are "finitary":

- PRA
- $I\Sigma_1$
- RCA_0, WKL_0
- $\widehat{PRA}^{\omega} + (QF{-}AC) + (WKL)$

The following theories are "arithmetic":

- PRA^{ω}
- PA
- ACA_0
- $HA^{\omega} + (AC) + (MP) + (WKL)$.
- $PA^{\omega} + (QF{-}AC) + (ACA)$

A finite-type variant of ACA_0

Let ACA_0^ω denote $PRA^\omega + (QF{-}AC) + (ACA)$.

This is a finite-type variant of ACA_0, and a conservative extension.

It is natural to consider an "arithmetic comprehension functional," μ:

$$\exists x\,(f(x) = 0) \to f(\mu(f)) = 0$$

for $f : \mathbb{N} \to \mathbb{N}$.

Theorem (Hunter). $ACA_0^\omega + (\mu)$ is a conservative extension of ACA_0.

A conservation theorem for measure theory

Code sets as characteristic functions, i.e. interpret $x \in Y$ as $\chi_Y(x) = 1$.

We can define unions and intersections. Using μ, we can also define countable unions and intersections.

Add a symbol λ for Lebesgue measure, and axioms:

- $\forall X \in \mathcal{P}(2^{\mathbb{N}}), \lambda(X) \geq 0$
- $\lambda(\emptyset) = 0$
- $\forall (X_n), (\forall i, j, j \neq i \to X_i \cap X_j = \emptyset) \to \lambda(\bigcup X_n) = \sum \lambda(X_n)$
- $\forall \sigma \in 2^{<\mathbb{N}}, \lambda([\sigma]) = 2^{-lth(\sigma)}$

In other words, λ is a measure on *all* subsets of $2^{\mathbb{N}}$.

A conservation theorem for measure theory

Theorem (Kreuzer). $ACA_0^\omega + (\mu) + (\lambda)$ is a Π^1_2-conservative extension of ACA_0.

The proof uses the Dialectica interpretation, together with delicate normalization arguments.