Notes on Π^1_1-conservativity, ω-submodels, and the collection schema*

Jeremy Avigad

January 8, 2002

Abstract

These are some minor notes and observations related to a paper by Cholak, Jockusch, and Slaman [3]. In particular, if T_1 and T_2 are theories in the language of second-order arithmetic and T_2 is Π^1_1 conservative over T_1, it is not necessarily the case that every countable model of T_1 is an ω-submodel of a countable model of T_2; this answers a question posed in [3]. On the other hand, for $n \geq 1$, every countable ω-model of $I\Sigma_n$ (resp. $B\Sigma_{n+1}$) *is* an ω-submodel of a countable model of $WKL_0 + I\Sigma_n$ (resp. $WKL_0 + B\Sigma_{n+1}$).

1 Π^1_1-conservativity and ω-submodels

If T is a theory in the language of second-order arithmetic, a Henkin model $\mathcal{M}$ of T can be viewed as a structure $\langle M, S_M, \ldots \rangle$, where first-order variables are taken to range over M, and second-order variables are taken to range over some subset S_M of the power set of M. If $M = \omega$ and $\mathcal{M}$ has the standard interpretations of $+$, $\times$, etc., then $\mathcal{M}$ is said to be an ω-model. If $\mathcal{M}_1 = \langle M_1, S_{M_1}, \ldots \rangle$ and $\mathcal{M}_2 = \langle M_2, S_{M_2}, \ldots \rangle$ are models, then $\mathcal{M}_1$ is said to be an ω-submodel of $\mathcal{M}_2$ if $M_1 = M_2$ and $S_{M_1} \subseteq S_{M_2}$ (note that M_1 and M_2 need not be ω!).

The theories RCA_0, WKL_0, ACA_0 are fragments of second-order arithmetic in which induction is restricted to Σ^0_1 formulae with parameters, and in which comprehension is replaced by recursive comprehension, a weak version of König's lemma, or arithmetic comprehension, respectively. From here on the general reference for subystems of second-order arithmetic is Simpson [10].

It is not hard to see that if T_1 and T_2 are theories in the language of second-order arithmetic and every countable ω-model of T_1 is an ω-submodel of a countable model of T_2, then T_2 is Π^1_1-conservative over T_1: if ψ is Π_1 and T_1 does not prove ψ, let $\mathcal{M}_1$ be a countable model of $T_1 + \neg\psi$; find a model $\mathcal{M}_2$ of T_2 such that $\mathcal{M}_1$ is an ω-submodel of $\mathcal{M}_2$; then $\mathcal{M}_2$ is a model of $T_2 + \neg\psi$.

*Carnegie Mellon Technical Report CMU-PHIL-125. Section 3 modified slightly, January 8, 2002.

Question 13.3 of [3] asks if the converse holds, i.e. whether the Π^1_1 conservation of T_2 over T_1 implies that every countable ω-model of T_1 is an ω-submodel of a model of T_2. The following proposition shows that the answer is no.

Proposition 1.1 *There is a sentence θ such that*

- *$ACA_0 + \theta$ is Π^1_1 conservative over ACA_0.*
- *$ACA_0 + \theta$ has no ω-model.*

Proof. In ACA_0 one has a Σ^1_1 truth predicate for Σ^1_1 sentences. Use this and the fixed-point lemma to construct a sentence ψ that says

"I am not provable from ACA_0 together with any true Σ^1_1 sentence."

Then $\neg\psi$ is a false Σ^1_1 sentence, and so has no ω-model. It suffices to show that $ACA_0 + \neg\psi$ is Π^1_1 conservative over ACA_0.

Suppose $ACA_0 + \neg\psi$ proves η, where η is Π^1_1. Then $ACA_0 + \neg\eta$ proves ψ. But then $ACA_0 + \neg\eta$ proves

"$\neg\eta$ is a true Σ^1_1 sentence and there is a proof of ψ from $\neg\eta$."

In other words, $ACA_0 + \neg\eta$ proves $\neg\psi$ as well as ψ. So $ACA_0 + \neg\eta$ is inconsistent, and hence ACA_0 proves η. $\square$

Something like the trick above (or, more precisely, the more refined version in the proof of Proposition 1.2) has been used recently by Arana [1]. The sentence ψ is equivalent to the assertion "any Π^1_1 consequence of ACA_0 together with my negation is true," and the proof above could equally well have been expressed with this formulation.

In the proof of Proposition 1.1, arithmetic comprehension was used to obtain an adequate definition of truth for Σ^1_1 formulae. I do not know whether it is possible to replace ACA_0 by RCA_0 in Proposition 1.1.

Question 13.4 of [3] asks whether the following holds: if T_0 and T_1 are Π^1_2 theories which are Π^1_1 conservative over a theory T, then $T_0 + T_1$ is necessarily Π^1_1 conservative over T. I suspect that answer is no, but the best I can come up with is the following "near miss."

Proposition 1.2 *There is a sentence θ such that*

- *θ is Π^1_2*
- *$\neg\theta$ can be put in the form $\exists n\ \forall X\ \exists Y\ \eta$, where η is arithmetic; in other words, $\neg\theta$ can be expressed with an existential number quantifier followed by a Π^1_2 sentence.*
- *$ACA_0 + \theta$ is Π^1_1 conservative over ACA_0*
- *$ACA_0 + \neg\theta$ is Π^1_1 conservative over ACA_0*

Proof. Do the Rosser trick: let θ say

> "If I am provable from ACA_0 plus a true Σ^1_1 sentence, then there is a shorter proof of my negation from ACA_0 plus a true Σ^1_1 sentence."

Here "shorter" really means "with smaller Gödel number." Then θ is of the form $\forall n(\exists X\ \alpha \to \exists Y\ \beta)$ with α and β arithmetic, and bringing quantifiers to the front in different orders allows one to put θ and $\neg\theta$ in the required forms. An argument similar to the one above shows that both $ACA_0 + \theta$ and $ACA_0 + \neg\theta$ are Π^1_1 conservative over ACA_0. For example, suppose d is a proof of a Π^1_1 sentence η in $ACA_0 + \theta$. Let d' be the corresponding proof of $\neg\theta$ in $ACA_0 + \neg\eta$, and let $e_0, \ldots, e_k$ enumerate all proofs of length less than d' of θ from Σ^1_1 sentences, say $\eta_0, \ldots, \eta_k$ respectively.

Now argue in ACA_0 to show η. First, if θ holds, we are done, using d. Otherwise, suppose $\neg\theta$. Then

> There is a proof of θ in ACA_0 together with a true Σ^1_1 sentence, and no shorter proof of of $\neg\theta$ in ACA_0 together with a true Σ^1_1 sentence.

If one of the η_i is true, then the corresponding proof, e_i, shows that θ is true, contradicting the assumption $\neg\theta$. So there is no proof of θ from a ACA_0 together with a true Σ^1_1 sentence with a proof shorter than d'. By the displayed assertion above, d' cannot be a proof of $\neg\theta$ from a true Σ^1_1 sentence; in other words, $\neg\eta$ is false, as required. $\square$

Note that in the presence of the Σ^1_2 axiom of choice, the number quantifier can be moved inwards, and $\neg\theta$ is equivalent to a Π^1_2 sentence. The problem is that Σ^1_1-AC is not a Π^1_2 axiom. But if the question above is rephrased so that T is not required to be Π^1_2 (i.e T_0 and T_1 are required to prove the same Π^1_2 sentences as T, but not necessarily extend T), this last observation yields a negative answer.

I do not know the answer to question 13.4 of [3] if one requires the theories involved to be true (in the standard model). In particular, for the θ used in the proof of Proposition 1.2, is it the case that every countable model of ACA_0 is an ω-submodel of a countable model of $ACA_0 + \theta$?

2 ω-models of Weak König's Lemma

In the 1980's, Harvey Friedman proved

Theorem 2.1 *WKL_0 is conservative over PRA for Π_2 sentences.*

Harrington later used a forcing argument, based on Jockusch and Soare's low basis theorem, to strengthen this to

Theorem 2.2 *WKL_0 is conservative over RCA_0 for Π^1_1 sentences.*

Friedman's theorem follows from this, since RCA_0 is easily interpreted in the fragment of first-order arithmetic $I\Sigma_1$, and an old theorem due to Mints, Parsons, and Takeuti independently shows that $I\Sigma_1$ is Π_2 conservative over PRA.

In [5], Hájek provides the following strengthening:

Theorem 2.3 *For all $n \geq 1$, $WKL_0 + I\Sigma_n$ is Π^1_1 conservative over $RCA_0 + I\Sigma_n$.*

In fact, Hájek obtains an *interpretation* of $WKL_0 + I\Sigma_n$, by formalizing the a recursion-theoretic construction of an ω-model of WKL_0 in $I\Sigma_n$. Avigad [2] independently internalized Harrington's forcing argument to obtain such an interpretation, for $n = 1$; Theorem 2.2 can then be obtained by relativizing the argument to the recursive sets (in which case, for $n \geq 2$, forcing for Π^0_n sentences is Π^0_n, and strong forcing for Σ^0_n sentences is Σ^0_n).

The authors of [3] note that methods of both Hájek [5] or Avigad [2] can be used to strengthen this result as follows:

Theorem 2.4 *For all $n \geq 1$, every countable ω-model of $RCA_0 + I\Sigma_n$ is an ω-submodel of a countable ω-model of $WKL_0 + I\Sigma_n$.*

This is correct, but a few more words of explanation are needed. Roughly, the problem is that Hájek's argument relativizes to a single set, but, on the surface, does not allow one to recapture the entire universe; my argument allows one to recapture the entire universe, but does not immediately work for $n \geq 1$ unless one restricts set quantifiers to the recursive sets.

The solution is to combine the two arguments. One fairly straightforward way to do this is to use a generic iteration as in [2] to capture all the sets of the original model, but restrict the forcing to recursive subsets of a single set in each step of the iteration. I will describe another method, which provides, in addition, Theorem 2.7 below. The idea is that Hájek's argument would work if one had a function explicitly enumerating the universe of sets; but one can simply force to add such a function.

Note that sequences of sets $\langle X_0, \ldots, X_k \rangle$ can be coded as a single set; I will assume that a reasonable coding has been chosen, so that the length of a sequence is unambiguous, and relevant properties can be verified in RCA_0. The notation $F(i) \supseteq F(j)$ in clause 2 of the following proposition should be read as "the sequence coded by $F(i)$ extends the sequence coded by $F(j)$."

Proposition 2.5 *Let $\mathcal{M}$ be any countable structure for the language of second-order arithmetic. Then one can expand $\mathcal{M}$ to a structure for a language with a new function symbol F denoting a function from M to S_M, such that the expanded structure satisfies the following:*

- $\forall i \; (\mathit{length}(F(i)) = i)$
- $\forall i, j \; (i > j \rightarrow F(i) \supseteq F(j))$
- $\forall X \; \exists i \; \exists j < i \; (F(i)_j = X)$

Proof. Take a *condition* to be a finite sequence of sets $\langle X_0, \ldots, X_k \rangle$, where "stronger than" means "extends, as a sequence." Then one can read off a function F from a suitably generic set. The properties that need to be met are: given $X \in S_M$, there is a condition in the generic that has X as an element; and given $n \in M$, there is a condition in the generic of length greater than or equal to n. $\square$

Note that the expanded model need not satisfy induction or comprehension for formulae involving F; so the interaction with F is mediated solely by the axioms above.

Proof of Theorem 2.4. Hájek shows how to construct a formula $\psi(x, y)$ so that the collection of sets of the form $S_x = \{x \mid \psi(x, y)\}$, as y ranges over $\mathbb{N}$, yields an omega model of $\mathit{WKL_0} + I\Sigma_n$, provably in $I\Sigma_n$. In other words, $I\Sigma_n$ proves that each axiom of $\mathit{WKL_0} + I\Sigma_n$ holds when one interprets second-order quantifiers as ranging over $\mathcal{S} = \{S_x \mid x \in \mathbb{N}\}$. The construction can be relativized to a set parameter Z.

With only slight modification, one can obtain a formula $\psi'(x, y, W)$, such that whenever W codes a finite sequence of sets, $\psi'(x, y, W)$ is defines an ω-model of $\mathit{WKL_0} + I\Sigma_n$ containing each of these sets; and with the property that if W' codes a sequence extending W, the ω-model containing W' includes the ω-model containing W. In other words, there is a formula $\psi'(x, y, W)$ such that $I\Sigma_n$ proves

- each axiom of $\mathit{WKL_0} + I\Sigma_n$ holds in the ω-model defined by $\psi'(x, y, W)$;
- $\forall \langle X_0, \ldots, X_{k-1} \rangle \; \forall i < k \; \exists j \; (\{x \mid \psi'(x, j, \langle X_0, \ldots, X_{k-1} \rangle)\} = X_i)$;
- $\langle X_0, \ldots, X_k \rangle \subseteq \langle Y_0, \ldots, Y_l \rangle \rightarrow \forall i \; \exists j \; (\{x \mid \psi'(x, i, \langle X_0, \ldots, X_{k-1} \rangle)\} = \{x \mid \psi'(x, j, \langle Y_0, \ldots, Y_{l-1} \rangle)\}$.

Now, given a countable model $\mathcal{M}$ of $\mathit{RCA_0} + I\Sigma_n$, expand it to a model $\mathcal{M}'$ of the three additional axioms in Proposition 2.5. Define $\theta(x, y)$ to be the formula

$$\psi'(x, (y)_0, F((y)_1)).$$

Intuitively, this represents the union of the ω-models defined by $\psi'(x, y, F(z))$ as z ranges over $\mathbb{N}$, so it is not hard to see that $\theta(x, y)$ defines an ω-model of $\mathit{WKL_0} + I\Sigma_n$ in $\mathcal{M}'$. In other words, letting

$$\mathcal{M}'' = \langle M, \{\{a \in M \mid \mathcal{M}' \models \theta(\bar{a}, \bar{b})\} \mid b \in M\}, \ldots \rangle$$

yields a countable model of $\mathit{WKL_0} + I\Sigma_n$ with the same first-order part as $\mathcal{M}$. $\square$

In fact, Hájek notes explicitly that for $n \geq 1$, the schema of Σ_n collection, denoted $B\Sigma_n$, can be used to justify Σ_n collection in the ω-model defined by ψ. As a result, his construction also shows

Theorem 2.6 *For all $n \geq 2$, $WKL_0 + B\Sigma_n$ is Π^1_1 conservative over $RCA_0 + B\Sigma_n$.*

Combining this with the argument above yields:

Theorem 2.7 *For all $n \geq 2$, every countable ω-model of $RCA_0 + B\Sigma_n$ is an ω-submodel of a countable ω-model of $WKL_0 + B\Sigma_n$.*

I should note that I do not know how to obtain these last two theorems using the methods of [2], or any other way; Hájek's formalization of the recursion-theoretic argument seems essential. I should also note that although it requires more work to obtain the model-theoretic results from the syntactic arguments in [2] and [5], the latter methods have the advantage of providing explicit translations between the theories, with polynomial bounds on increase in proof length. Finally, Simpson and Smith [11] shows that Theorem 2.7 holds for $n = 1$ if one adds an axiom asserting that exponentiation is total. For related results in the context of bounded arithmetic, see Ferreira [4].

3 Separating Σ_{n+1} collection and Σ_n induction

In the language of first-order arithmetic, the Σ_n collection schema, $B\Sigma_n$, is as follows:

$$\forall a, \vec{z}\ (\forall x < a\ \exists y\ \theta(x, y, \vec{z}) \rightarrow \exists b\ \forall x < a\ \exists y < b\ \theta(x, y, \vec{z}))$$

where θ is Σ_n. Below $B\Sigma_n$ is also used to denote the fragment of arithmetic in which induction is replaced by the schema above.

The following theorem is due to Friedman and Paris, independently:

Theorem 3.1 *For each $n \geq 0$, Σ_{n+1} collection is Π_{n+2} conservative over $I\Sigma_n$.*

Paris and Wilkie showed in [8]:

Theorem 3.2 *For each $n \geq 0$, $I\Sigma_n$ does not prove Σ_{n+1} collection.*

In [7, page 331], Paris notes that one can extract the following from the proof:

Theorem 3.3 *For each $n \geq 0$, there is a Σ_{n+2} sentence provable from $B\Sigma_{n+1}$ but not $I\Sigma_n$.*

Finding the Σ_{n+2} sentence takes some digging, however, and seems to require a trick (used in Chapter IV of [6]), as follows.

First, note by Gödel's incompleteness theorem, there is a Σ_0 formula $\psi(x)$ such that $\exists x\ \psi(x)$ is false, but consistent with Peano arithmetic. So, in any model, an element a satisfying $\psi(a)$ is necessarily nonstandard. We can choose ψ so that $I\Sigma_0$ proves $\forall x, y\ (\psi(x) \wedge \psi(y) \rightarrow x = y)$.

Let $\alpha(e, x)$ say, roughly, "e is a Σ_{n+1} formula defining x," using a Σ_{n+1} truth predicate, as in the Paris-Wilkie proof. Let $\alpha(e, x)$ be equivalent to $\exists u\ \theta(e, u, x)$, where θ is Π_n.

The sentence I extracted from the Paris-Wilkie proof is

$$\exists b \, \forall a \, (\psi(a) \land \forall x < (a+1) \, \exists e < a \, \exists u \, \theta(e,u,x)) \to \\ \forall x < (a+1) \, \exists e < a \, \exists u < b \, \theta(e,u,x))$$

Notice that if you move the $\exists b$ to the consequence of the implication, this just says that a certain instance of collection holds for $a+1$, where a is the value satisfying ψ, if there is one. It is false in the Paris-Wilkie model K_{n+1} of $I\Sigma_n$, assuming the nonstandard model has an element satisfying ψ; but it is an easy consequence of Σ_{n+1} collection.

Bringing the universal quantifier over x out of the antecedent yields

$$\exists b \, \forall a \, \exists x \, (\psi(a) \land (x < (a+1) \to \exists e < a \, \exists u \, \theta(e,u,x))) \to \\ \forall x < (a+1) \, \exists e < a \, \exists u < b \, \theta(e,u,x))$$

Using the fact that there is at most one a satisfying $\psi(a)$, we can switch the order of the second and third quantifiers:

$$\exists b \, \exists x \, \forall a \, (\psi(a) \land (x < (a+1) \to \exists e < a \, \exists u \, \theta(e,u,x))) \to \\ \forall x < (a+1) \, \exists e < a \, \exists u < b \, \theta(e,u,x))$$

Using Σ_n collection, $I\Sigma_n$ proves that this statement is Σ_{n+2}.

A different construction of a Σ_{n+2} sentence satisfying Theorem 3.2 is described in Section 4.1 of Sieg [9].

4 Some thoughts on RT^2_2 and $B\Sigma_2$

Let RT^2_2 denote the infinitary Ramsey's theorem for two-colorings of pairs of integers, as described in [3]. Hirst has shown that the infinitary Ramsey's theorem for pairs implies $B\Sigma_2$; on the other hand, [3] shows that the first-order consequences of $RCA_0 + RT^2_2$ are included in $I\Sigma_2$.

It is still open as to what exactly the first-order consequences of $RCA_0 + RT^2_2$ are, and nothing as of yet precludes the possibility that the first-order consequences of RT^2_2 are *exactly* those of $B\Sigma_2$. In other words, one might try to prove that $RCA_0 + B\Sigma_2 + RT^2_2$ is conservative over $B\Sigma_2$, by replacing $I\Sigma_2$ by $B\Sigma_2$ in Corollary 8.6, Lemma 9.4, and Lemma 10.4 of [3].

Two observations are encouraging in that regard:

- By the results of the Section 2, every countable model of $RCA_0 + B\Sigma_2$ is an ω-submodel of a countable model of $WKL_0 + B\Sigma_2$. (This is the analogue of Corollary 8.6.)

- The use of Σ_2 induction in the proof of Lemma 9.9 in [3] is unnecessary; the choice principle it is used to derive in fact follows from WKL_0 alone. (See [10], Lemma VIII.2.4, page 319.) So one use of $B\Sigma_2$ in the proof of Lemma 9.4 can be eliminated.

But I see no way of modifying the proof of Lemma 9.10 in [3] to obtain the analogue of Lemma 9.4 for $B\Sigma_2$, let alone the corresponding version of Lemma 10.4.

References

[1] Andrew Arana. Solovay's theorem cannot be simplified. To appear.

[2] Jeremy Avigad. Formalizing forcing arguments in subsystems of second-order arithmetic. *Annals of Pure and Applied Logic*, 82:165–191, 1996.

[3] Peter Cholak, Carl Jockusch, and Theodore Slaman. On the strength of ramsey's theorem for pairs. *Journal of Symbolic Logic*, 66:1–55, 2001.

[4] Fernando Ferreira. A feasible theory for analysis. *Journal of Symbolic Logic*, 59:1001–1011, 1994.

[5] Petr Hájek. Interpretability and fragments of arithmetic. In Peter Clote and Jan Krajıček, editors, *Arithmetic, Proof Theory, and Computational Complexity*, pages 185–196. Oxford University Press, Oxford, 1993.

[6] Petr Hájek and Pavel Pudlák. *Metamathematics of first-order arithmetic*. Springer, Berlin, 1993.

[7] Jeff Paris. A hierarchy of cuts in models of arithmetic. In Pacholski et al., editor, *Model Theory of Algebra and Arithmetic*, Lecture Notes in Mathematics 834, pages 312–337. Springer, 1980.

[8] Jeff Paris and Robin Kirby. Σ_n collection schemas in arithmetic. In Paris et al., editor, *Logic Colloquium '77*, pages 199–209. North Holland, 1978.

[9] Wilfried Sieg. Fragments of arithmetic. *Annals of Pure and Applied Logic*, 28:33–72, 1985.

[10] Stephen Simpson. *Subsystems of Second-Order Arithmetic*. Springer, Berlin, 1998.

[11] Stephen Simpson and Rick Smith. Factorization of polynomials and Σ^0_1 induction. *Annals of Pure and Applied Logic*, 31:289–306, 1986.