Forcing in proof theory*

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Abstract

Paul Cohen's method of forcing, together with Saul Kripke's related semantics for modal and intuitionistic logic, has had profound effects on a number of branches of mathematical logic, from set theory and model theory to constructive and categorical logic. Here, I argue that forcing also has a place in traditional Hilbert-style proof theory, where the goal is to formalize portions of ordinary mathematics in restricted axiomatic theories, and study those theories in constructive or syntactic terms. I will discuss the aspects of forcing that are useful in this respect, and some sample applications. The latter include ways of obtaining conservation results for classical and intuitionistic theories, interpreting classical theories in constructive ones, and constructivizing model-theoretic arguments.

1 Introduction

In 1963, Paul Cohen introduced the method of forcing to prove the independence of both the axiom of choice and the continuum hypothesis from Zermelo-Fraenkel set theory. It was not long before Saul Kripke noted a connection between forcing and his semantics for modal and intuitionistic logic, which had, in turn, appeared in a series of papers between 1959 and 1965. By 1965, Scott and Solovay had rephrased Cohen's forcing construction in terms of Boolean-valued models, foreshadowing deeper algebraic connections between forcing, Kripke semantics, and Grothendieck's notion of a topos of sheaves. In particular, Lawvere and Tierney were soon able to recast Cohen's original independence proofs as sheaf constructions.[1]

It is safe to say that these developments have had a profound impact on most branches of mathematical logic. These various disciplines, in return, provide a

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[1]A historical account of the development of forcing can be found in [61]. For the development of Kripke semantics, see [36]; for a historical overview of the connections between logic and sheaf theory, see the prologue to [55].

range of perspectives that can help us understand why forcing is such a powerful tool. For the set theorist, forcing provides a means of extending a model of set theory by adding a "generic" object, in such a way that truth in the extension is determined by approximations to the generic that live in the original model. From the point of view of modal logic, forcing provides a means by which we can explicate the notion of necessary truth, or truth in all possible worlds, in terms of local truth, or truth in individual worlds. Forcing also provides a semantics for intuitionistic logic based on a notion of partial information, or states of knowledge over time. For the recursion theorist, forcing provides a convenient way of describing constructions in which a sequence of requirements is satisfied one at a time (see, for example, [21, 56]). For the model theorist, forcing is a construction that provides a suitably generic model of any inductive ($\forall\exists$) theory. From the point of view of sheaf theory, forcing provides a way of describing the internal logic of a topos. For the descriptive set theorist, forcing provides a means of saying what it means for a property to be "generically true" of a Polish space (see, for example, [49]). The point of view of the effective descriptive set theorist, as in [68], lies somewhere between that of the descriptive set theorist and recursion theorist. Ideas from forcing have even been influential in computational complexity; for example, the separation of complexity classes relativized to an oracle (e.g. as in [10]) can often be viewed as resource-bounded versions of forcing.

So there you have it: insofar as diagonalization, modality, local and global notions of truth, and iterative constructions are central to mathematical logic, forcing offers something for everyone.

Perhaps the *only* branch of logic absent from this list is proof theory. When one thinks of proof theory, one usually thinks of formal deductive systems, cut elimination, normalization, ordinal analysis, and functional interpretation; forcing may be close to the last thing that comes to mind. The goal of this survey is, quite simply, to change this perception. In particular, my aim will be to characterize forcing from a syntactic point of view, and emphasize the features that make it useful from a proof-theoretic perspective. I will then present some proof-theoretic applications, by way of illustration.

Today, the phrase "proof theory" includes a variety of disciplines. Broadly construed, it describes the general study of formal deductive systems using mathematical methods. Even if we restrict our attention to formal theories that are most relevant to *mathematical* reasoning (like propositional logic, first-order logic, and higher-order logic), one can still identify a number of distinct subdisciplines. For example, in structural proof theory, the focus is on properties of a deductive system that depend on the precise way in which its rules are formulated, as well as transformations between proofs, normal forms, and search methods. In contrast, proof complexity is concerned with obtaining upper and lower bounds on lengths of proofs; from this perspective different deductive systems are viewed as equivalent when there are efficient translations between them. By way of clarification, what I am interested in here is the traditional, "metamathematical" branch of proof theory, where the goal is to understand various aspects of classical mathematics in syntactic, constructive, or otherwise explicit terms. From this point of view, what one is really interested in is the provability *relation*; the choice of a particular deductive system is relevant only insofar as it is useful to understanding this relation, in an explicit, finitary way.[2]

What, then, does forcing offer the traditional proof theorist? In a sense, much the same thing that it offers the set theorist: a powerful tool for "reducing" one axiomatic theory to another, or comparing the strength of two such theories. Many results in proof theory take the form of *conservation theorems*, which is to say, they amount to showing that for any sentence φ in a certain class Γ, if a theory T_1 proves φ, then an apparently weaker one, T_2, proves it as well (or perhaps a suitable translation, φ'). These include equiconsistency results, in the special case where φ is simply falsity, $\bot$. But note that even classical set-theoretic equiconsistency results typically yield more information; for example, Gödel's use of the constructible hierarchy shows that any Π^1_3 statement in the analytic hierarchy provable in Zermelo-Fraenkel set theory with the continuum hypothesis and the axiom of choice is, in fact, provable in Zermelo-Fraenkel set theory alone.

Though the comparison to set theory may be illuminating, there are important differences in emphasis. For one thing, proof theorists are typically interested in theories much weaker than full *ZFC*, which is to say, theories that more minimally suffice to capture ordinary mathematical arguments, and which one has a better chance of understanding in constructive terms. Second, of course, is the proof-theorist's emphasis on syntax. For the set theorist, forcing is a model-theoretic technique that happens to have a useful syntactic interpretation, whereas, for the proof theorist, the situation is reversed: the model-theoretic interpretation may have heuristic value, but may be otherwise irrelevant. Finally, there is the underlying logic: whereas set theorists typically restrict their attention to classical logic, proof theorists are keenly interested in constructive aspects of forcing as well.

The outline of this paper is as follows. In Section 2, I will describe various forcing relations from a proof-theoretic perspective. Then, in Sections 3–5, I will discuss some applications, trying to convey a general sense of the uses to which forcing can be put, without providing much detail.

This survey is neither comprehensive, nor even balanced. I ask the reader to keep in mind that, by focusing on examples with which I am most familiar, I am providing an inflated view of my own contributions to the subject. At the same time, I apologize to the many people whose work I have slighted.

[2]Although I will discuss forcing proofs of cut elimination, generally speaking, the uses of forcing I will describe are not closely tied to the particular specification of a deductive system; so it is not clear to me whether forcing can offer the kind of information that is generally of interest in structural proof theory. Forcing methods *have*, however, been important in proof complexity. Below, I will focus on the use of forcing in obtaining efficient interpretations between theories, and therefore upper bounds on the increase in length of proof. But there have been other, perhaps more striking, applications of forcing towards obtaining *lower* bounds, as in [82, 1, 67, 50, 78]. I will, regrettably, not discuss these methods here.

2 The forcing relation

2.1 Minimal, classical, and intuitionistic logic

Proof theorists commonly distinguish between three variants of first-order logic, namely, minimal, intuitionistic, and classical. To have a uniform basis for comparison, I will henceforth take the basic connectives to be $\wedge$, $\vee$, $\to$, $\forall$, $\exists$, and $\bot$, with $\neg\varphi$ defined as $\varphi \to \bot$.

Of the three types of logic, minimal logic is the fragment with the nicest computational interpretation. Formulae can be seen as datatype specifications of their own proofs: for example, a proof of $\varphi \wedge \psi$ can be viewed as an ordered pair, consisting of a proof of φ and a proof of ψ; a proof of $\varphi \to \psi$ can be viewed as a procedure transforming a proof of φ to a proof of ψ, and so on. Note that minimal logic has nothing to say about $\bot$, which is therefore treated as an arbitrary propositional variable. Intuitionistic logic adds the principle $\bot \to \varphi$, *ex falso sequitur quodlibet*, which is computationally palatable only because (we hope) there are no proofs of $\bot$. Classical logic can be obtained from minimal logic by adding either the principle of *double negation elimination*, $\neg\neg\varphi \to \varphi$, or the *law of the excluded middle*, $\varphi \vee \neg\varphi$.

Although the inclusions just indicated are proper, one can interpret both classical and intuitionistic logic in minimal logic. For intuitionistic logic, the following simple device works: if φ is any formula, let φ^* denote the result of replacing each atomic formula A with $A \vee \bot$. Then, trivially, $\bot \to A^*$ is derivable in minimal logic, yielding *ex falso* for atomic formulae; and it is not hard to show that the principle for arbitrary formulae follows from this. Let $\vdash_I$ and $\vdash_M$ denote intuitionistic and minimal provability, respectively, and if Γ is any set of sentences, let Γ^* denote $\{\psi^* \mid \psi \in \Gamma\}$. Then we have:

Proposition 2.1 *If* $\Gamma \vdash_I \varphi$*, then* $\Gamma^* \vdash_M \varphi^*$*.*

Replacing atomic formulae A with $\neg\neg A$ instead of $A \vee \bot$ would have worked just as well.

To interpret classical logic in minimal logic, one can use the *double-negation translation* due to Gödel and Gentzen. If φ is any formula, let φ^N denote the result of adding a double negation in front of atomic formulae, and in front of subformulae with outermost connectives $\exists$ and $\vee$. Clearly φ^N is classically equivalent to φ. By induction on formulae, one can show:

Lemma 2.2 *For each formula* φ*,* $\vdash_M \varphi^N \leftrightarrow \neg\neg\varphi^N$*.*

Then, using $\vdash_C$ to denote classical provability and Γ^N to denote $\{\psi^N \mid \psi \in \Gamma\}$, the following is obtained by induction on derivations:

Proposition 2.3 *If* $\Gamma \vdash_C \varphi$*, then* $\Gamma^N \vdash_M \varphi^N$*.*

Since, in minimal logic, $\neg\neg(\varphi \vee \psi)$ is equivalent to $\neg(\neg\varphi \wedge \neg\psi)$ and $\neg\neg\exists x\, \varphi$ is equivalent to $\neg\forall x\, \neg\varphi$, one can view the double-negation translation as, essentially, eliminating $\vee$ and $\exists$ altogether from classical formulae.

We will see below that Cohen's original "strong forcing" relation is best understood in terms of a slick variant of the double-negation translation known as the *Kuroda translation* [53]. For any formula φ, let φ^K denote the result of doubly-negating atomic formulae, and adding a double negation *after* each universal quantifier. Although φ^K is not always equivalent to φ^N in intuitionistic logic, it turns out that $\neg\neg\varphi^K$ is. Combining this fact with the previous proposition, and writing $\neg\neg\Gamma^K$ for $\{\neg\neg\psi^K \mid \psi \in \Gamma\}$, we have:

Proposition 2.4 *If* $\Gamma \vdash_C \varphi$*, then* $\neg\neg\Gamma^K \vdash_I \neg\neg\varphi^K$*.*

2.2 Kripke semantics and forcing

For simplicity, let us fix a first-order relational language, L, without equality. In the context of minimal logic, a Kripke structure for L consists of a tuple, $\langle P, D, \Vdash \rangle$, where:

- P is a poset, which one can think of as representing either possible worlds at different points in time, or states of partial knowledge;
- D is a function which assigns a set, $D(p)$ or "the domain at p," to each element p of the subset; and
- for each k-ary relation symbol R and each element p of the poset, $p \Vdash R(a_0, \ldots, a_{k-1})$ denotes a k-ary relation on $D(p)$.

In other words, D and $\Vdash$ taken together provide an ordinary first-order L-structure at each element p of the poset. The two are required to satisfy the following monotonicity conditions: if $q \leq p$, then

- $D(q) \supseteq D(p)$, and
- if $p \Vdash A(a_0, \ldots, a_{k-1})$ then $q \Vdash A(a_0, \ldots, a_{k-1})$.

Think of $q \leq p$ as asserting that q is stronger than p, in that it provides more information, or corresponds to a later point in time. The monotonicity clauses then assert that when one passes to a stronger condition (or a later point in time), more elements of the domain become visible, and more atomic facts are seen to be true. (The use of $\leq$ rather than $\geq$ to denote "stronger than" accords well with algebraic interpretations of forcing, and has become almost standard.)

Let $L(D)$ denote the extension of L to a language with extra constants to denote elements of the sets $D(p)$. Reading the definition of a Kripke structure above as defining the notion of forcing for atomic sentences of $L(D)$, one extends the relation to the whole of $L(D)$ by induction on formulae:

1. $p \Vdash \theta \wedge \eta$ if and only if $p \Vdash \theta$ and $p \Vdash \eta$
2. $p \Vdash \theta \vee \eta$ if and only if $p \Vdash \theta$ or $p \Vdash \eta$
3. $p \Vdash \theta \rightarrow \eta$ if and only if $\forall q \leq p \; (q \Vdash \theta \rightarrow q \Vdash \eta)$

4. $p \Vdash \forall x\, \varphi(x)$ if and only if $\forall q \leq p\, \forall a \in D(q)\, (q \Vdash \varphi(a))$
5. $p \Vdash \exists x\, \varphi(x)$ if and only if $\exists a \in D(p)\, (p \Vdash \varphi(a))$

One can easily show that monotonicity extends to the entire language, and that the interpretation is sound for minimal logic:

Proposition 2.5 *Let $\Vdash_M$ denote any forcing relation obtained as above.*

1. *(monotonicity) For every p and q, $p \Vdash_I \varphi$ and $q \leq p$ imply $q \Vdash_I \varphi$.*
2. *(soundness) For every φ, $\vdash_M \varphi$ implies $\Vdash_M \varphi$.*

Perhaps the best way to understand the forcing definition is to view it as the result of making the fewest changes possible to classical semantics in order to get monotonicity to hold. For $\wedge$, $\vee$, and $\exists$ there is nothing to be done; only $\rightarrow$ and $\forall$ require some thought. Defining $p \Vdash \varphi \rightarrow \psi$ as $(p \Vdash \varphi) \rightarrow (p \Vdash \psi)$, for example, would not work, since φ may be false at p but may become true at a later stage. The forcing clause for implication, and similarly for the universal quantifier, simply takes into account what may happen later on. Thus, given that one is committed to an interpretation of minimal logic based on a notion of partial information, the clauses above almost write themselves.

The notation $\Vdash \varphi$, read "φ is forced," means that every element of the poset forces φ. By monotonicity, if there is a least element $\emptyset$ in the poset, this is equivalent to saying $\emptyset \Vdash \varphi$. From a semantic point of view, it is nice to know that Kripke semantics is complete for minimal logic; in fact, there is a single "universal" model such that the formulae that are forced are exactly the ones that are valid. But completeness is of less interest to the proof theorist, who is typically more interested in *specific* interpretations of axiomatic theories.

The most straightforward way to extend the semantics to intuitionistic logic is simply to declare that $\bot$ is interpreted as falsity at each node. In other words, one requires the intuitionistic forcing relation $\Vdash_I$ to satisfy the following clause:

- $p \nVdash_I \bot$

Then monotonicity is preserved, and we can show that $\bot \rightarrow \varphi$ is forced for every φ. So, we have the following:

Proposition 2.6 1. *(monotonicity) For every p and q, $p \Vdash_M \varphi$ and $q \leq p$ imply $q \Vdash_M \varphi$.*

2. *(soundness) For every φ, $\vdash_I \varphi$ implies $\Vdash_I \varphi$.*

One can extend forcing semantics to classical logic via the double-negation translations of classical logic to minimal logic. For example, using the Gödel-Gentzen translation, we can define a classical forcing relation, $\Vdash_C \varphi$, by $\Vdash_M \varphi^N$. Then, immediately, from properties of the double-negation translation and forcing for minimal logic, we have:

Proposition 2.7 1. *(monotonicity) $p \Vdash_C \varphi$ and $q \leq p$ imply $q \Vdash_C \varphi$.*

2. *(soundness)* $\vdash_C \varphi$ *implies* $\Vdash_C \varphi$.

3. *(genericity)* $p \Vdash_C \varphi$ *if and only if* $\forall q \leq p \, \exists r \leq q \, (r \Vdash_C \varphi)$.

The right hand side of the equivalence in the last clause is just the assertion $p \Vdash_C \neg\neg\varphi$, and is commonly read "$\varphi$ is generically valid below p." Unwrapping the double-negation translation and the definition of forcing for minimal logic, we can obtain a more direct, and perhaps more familiar, definition of the classical forcing relation. For example, we have

- $p \Vdash_C \neg\theta$ if and only if $\forall q \leq p \, (q \nVdash_C \theta)$
- $p \Vdash_C \theta \vee \eta$ if and only if $\forall q \leq p \, \exists r \leq q \, ((r \Vdash_C \theta) \vee (r \Vdash_C \eta))$
- $p \Vdash_C \exists x \, \theta(x)$ if and only if $\forall q \leq p \, \exists r \leq q \, \exists a \in D(r) \, (r \Vdash_C \theta(a))$

The soundness clause in the previous proposition easily implies the following strengthening: if $\Gamma \vdash_C \varphi$, and p forces every sentence in Γ, then p forces φ.

The classical forcing relation I have just described is sometimes known as *weak forcing*. Using the Kuroda translation, we can define an alternative notion of *strong forcing*, $\Vdash_S \varphi$, by $\Vdash_M \varphi^K$. Then by the properties of the Kuroda translation we have

Proposition 2.8 1. $\Vdash_C \varphi$ *if and only if* $\Vdash_S \neg\neg\varphi$.

2. *Suppose* Γ *is any set of sentences and* $\Gamma \vdash_C \varphi$. *Then if every sentence in* Γ *is generically valid below* p, *so is* φ.

So weak forcing can be defined in terms of strong forcing, and the latter is often useful in contexts where one wants to keep the complexity of the forcing notions low.

2.3 Variations

In this section I would like to catalogue a number of variations on the basic forcing relations described above. The reader may find the list tedious, so I recommend skimming it and referring back to it as necessary.

The description of Kripke semantics above was limited to relational languages without equality. But it is easy to extend the semantics to languages with both function symbols and equality. Moreover, Kripke semantics offers natural ways of modeling logics where terms are only partially defined, which is to say, they may fail to denote existing objects. For extensions like these, see [80].

In passing from minimal to intuitionistic logic, we added the clause

- $p \nVdash \bot$

But all we really need is that $\bot \to \varphi$ is forced, and, furthermore, it suffices to make sure that this is the case when φ is atomic. We therefore obtain a more general class of Kripke structures by replacing the clause above with

- if $p \Vdash \bot$ then $p \Vdash A$, for every atomic formula A.

One can then show inductively that whenever $p \Vdash \bot$, p forces every formula φ. So, the next effect is that we are allowing a region of the poset, closed downwards, at which everything becomes true. These are sometimes known as "exploding Kripke models." Since any such model can be transformed into a regular Kripke model by simply cutting away the inconsistent part, it is hard to believe that this idea can be useful. But it can: it often helps in carrying out constructions in weak or constructive theories, since, for a given description of the model, there may be no effective way of testing whether or not a node is consistent.

When it comes to minimal and intuitionistic logic, one can loosen up the clauses for $\vee$ and $\exists$. For example, a Beth model is essentially a Kripke model in which the underlying poset is a tree. In such a model, a set of nodes C is said to *cover* a node p if every maximal branch passing through p also passes through an element of C. In a Beth model, one weakens the clauses for $\vee$ and $\exists$ as follows:

- $p \Vdash \varphi \vee \psi$ if and only if there is a covering C of p, such that for every $q \in C$, $q \Vdash \varphi$ or $q \Vdash \psi$
- $p \Vdash \exists x\ \varphi(x)$ if and only if there is a covering C of p, such that for every $q \in C$ there is an $a \in D(q)$ such that $q \Vdash \varphi(a)$

In other words, p forces $\varphi \vee \psi$ if and only if at p one can say with confidence that one of the disjuncts will eventually become true; and similary for $\exists x\ \varphi$.

One can also generalize the semantics by allowing an arbitrary category in place of the underlying poset. If p and q are elements of the category, one can think of an arrow from p to q as denoting that p is stronger than q, so Kripke models over a poset are a special case of this semantics. For each arrow f from p to q one needs more generally a translation function $F(f)$ from the domain at q to the domain at p; both the domains and the interpretations of the relations symbols of the underlying language have to satisfy the natural generalizations of the monotonicity conditions for Kripke models. Such structures are usually called *presheaf models*.

Moreover, the sheaf-theoretic notion of a *Grothendieck topology* can be understood as a generalization of the covering notion for Beth models to presheaf models. A presheaf model equipped with a Grothendieck topology (and satisfying a condition that asserts, roughly, that the existence of elements in the various domains is compatible with the notion of covering) is called a *sheaf model*. Such structures can be used to intepret not only first-order logic, but higher-order logic as well, in a natural way. (Most of the semantic variations considered in this section are discussed in [80]. For sheaf models in particular, see [55, 37].)

Covering notions are typically less relevant to classical logic, because there $\vee$ and $\exists$ can be defined in terms of $\wedge$ and $\forall$. But, indeed, there is a sense in which they are unnecessary even for intuitionistic and minimal logic; after all, Kripke semantics itself is complete for these. The point is that as one generalizes the semantics, one has more more flexibility in building models, making it easier to interpret the constructions in weak or restricted theories. For example, with the wider classes of models, completeness proofs become almost trivial; see, for example, the syntactic sites in [55], Friedman's construction of Beth models in [80], or constructions of models of first-order theories in [28, 64, 9]. This idea will be developed a bit further in Section 5, where forcing will be seen to provide a kind of "poor man's model theory."

Of course, it is the classical version of forcing that is essentially the notion that set theorists know and love. In standard set-theoretic constructions (see e.g. [52, 71]) sets in the generic extension are named by elements of the ground model, in such a way that the relations of elementhood and identity are settled by the generic. View these names as the inhabitants of the world associated to each partial condition, so $p \Vdash x \in y$, for example, means that $x \in y$ becomes true at p. The central property of the generic — i.e. the fact that it meets every dense definable set of conditions — translates exactly to the truth conditions on formulae deriving from properties of the double-negation translation.

The insight coming from the Scott-Solovay approach is that one can turn the relation around, and view the forcing relation as assigning, to each formula φ, an evaluation $[\![\varphi]\!]$ in a suitable algebraic structure. If the poset P itself forms a complete Boolean or Heyting algebra, then for the classical and intuitionistic versions of forcing, respectively, we can take

$$[\![\varphi]\!] = \bigvee \{p \mid p \Vdash \varphi\}.$$

More generally, formulas have to be evaluated in a suitable completion of P. For example, with an intuitionistic forcing relation, the assignment

$$[\![\varphi]\!] = \{p \mid p \Vdash \varphi\}$$

yields values in the complete Heyting algebra of downwards-closed subsets of P; and with a classical (weak) forcing relation, it yields values in the complete Boolean algebra of regular open subsets of P, where the topology is given by the basis of sets of the form $B_p = \{q \mid q \leq p\}$. Similarly, higher-order intuitionistic forcing relations can be evaluated in the topos of sheaves over P. So, in addition to the model-theoretic and syntactic views of forcing, there are algebraic and topological views of forcing as well. See [37, 52, 55, 80] for details on these points of view.

I have not even touched on the use of Kripke structures to model the semantics of various modal operators. For this, see, for example, [18, 38, 44].

2.4 The syntactic perspective

Up to this point, I have been discussing forcing semantics as though the underlying Kripke structures live "in the real world." But from a hard core proof-theoretic point of view, there is no "real world," beyond syntax. In other words, the only way to understand Kripke or forcing semantics for an axiomatic theory, T_1, is in terms of another axiomatic theory, T_2.

Of course, one way to do this is to choose a theory T_2, like ZFC, that suffices to formalize ordinary mathematical arguments, and view the model-theoretic constructions as taking place there. Kripke structures and generic models are mathematical objects like any other, and the associated semantic notions can be defined by recursions on terms and formulae in the usual way.

One can proceed more frugally, however, by *interpreting* the relevant Kripke structure (or generic model) in T_2. In other words, one can define predicates $Cond$, $\leq$, and $Name$ in the language of T_2, intended to denote the conditions, the ordering, and the elements of the various domains. Then, for each relation symbol $A(x_0, \ldots, x_{k-1})$ in the language of T_1, one defines, in T_2, a relation $p \Vdash A(a_0, \ldots, a_{k-1})$ on conditions and names. The inductive forcing clauses then provide a translation from formulae θ in the language of T_1 to formulae $p \Vdash \theta$ in the language of T_2. To complete the interpretation, one need only show, in T_2, that the axioms of T_1 are forced, and that forcing respects the logic of T_2.

All this implies that whenever T_1 proves a formula φ, T_2 proves that φ is forced. Assuming T_2 proves that $\bot$ is not forced, this is enough to show that T_1 is consistent relative to T_2. But often it will be the case that T_2 can show that for some class Γ of formulae φ, $\Vdash \varphi$ is equivalent to φ. The interpretation then shows that T_2 is conservative over T_1 for formulae in Γ.

When it comes to handling the underlying logic, something interesting happens. Assuming that T_2 can verify the basic properties of the ordering and forcing relation for atomic formulae (i.e. transitivity, monotonicity, etc.), minimal logic suffices to prove that minimal logic is forced, for the minimal version of the forcing relation; intuitionistic logic suffices to prove that intuitionistic logic is forced, for the intuitionistic version of the forcing relation; and classical logic suffices to prove that classical logic is forced, for the classical version of the forcing relation. Thus, the various forcing relations are well suited to interpretations that do not cross logical boundaries.

Moreover, suitable variations of the forcing relation can also be used to interpret classical logic in intuitionistic logic, or classical and intuitionistic logic in minimal logic. For example, using the "exploding" Kripke semantics described in the last section, minimal logic suffices to interpret the intuitionistic forcing relation. And if enough double-negations are kept around, minimal logic can even verify that classical logic is forced, under the classical forcing relation. The latter requires, for example, interpreting $\Vdash \neg\neg\varphi$ as $\forall p \, \neg \forall q \leq p \, \neg(q \Vdash \varphi)$ instead of the usual notion of generic validity, $\forall p \, \exists q \leq p \, (q \Vdash \varphi)$. We will see below that in many applications, this is sufficient. In some cases, however, it is useful to be able to use the more common form of genericity in an intuitionistic setting. Beeson [13] presents a version of the forcing relation that is classically but not intuitionistically equivalent to the classical forcing relation; his version has the property that the generic validity of φ is expressed as $\forall p \, \exists q \leq p \, (q \Vdash \varphi)$, and yet the validity of intuitionistic logic under the forcing relation can be demonstrated intuitionistically.[3]

In the sections below, I will consider ways in which forcing methods can be used to prove conservation results. I will not limit myself to the proof-theoretic perspective, which is to say, I will not hesitate to mention model-theoretic, recursion theoretic, and algebraic constructions as well. But my emphasis will be on instances where these constructions are relevant to obtaining syntactic translations, in the manner I have just described.

Summing up, to interpret a theory T_1 in T_2, one can follow this general pattern:

1. Define a poset and appropriate forcing notions in T_2.
2. Show, in T_2, that the axioms of T_1 are forced.
3. Conclude that if T_1 proves φ, then T_2 proves "φ is forced."
4. For partial conservativity, show that for formulae φ in an appropriate class Γ, if T_2 proves "φ is forced," then T_2 proves φ.

3 Subsystems of second-order arithmetic

Let us think of the language of second-order arithmetic as a two-sorted first-order language, with variables $x, y, z, \ldots$ ranging over numbers, and variables $X, Y, Z, \ldots$ ranging over sets of numbers. We can take the language to have symbols $0, 1, +, \times, <$, as well as a symbol $\in$ relating the two sorts. As an axiomatic theory, full second-order arithmetic is given by (the universal closures of) the following axioms:

- quantifier-free defining equations for the basic symbols of arithmetic
- the full schema of comprehension: $\exists Z \, \forall x \, (x \in Z \leftrightarrow \varphi)$ for each formula φ in which Z is not free
- induction on the natural numbers: $0 \in Y \land \forall x \, (x \in Y \rightarrow x + 1 \in Y) \rightarrow \forall x \, (x \in Y)$

Note that by using the comprehension schema and the axiom of induction in tandem, one obtains the schema of induction for arbitrary formulae φ. Of course, no effective set of axioms is complete for truth in the standard model; one can also consider, for example, various choice principles in the context of second-order arithmetic. (It is folklore that the full choice schema, and even a stronger schema of dependent choice, is interpretable in second-order arithmetic, by developing Gödel's constructible hierarchy there. See [72, 57] and the discussion in Sections 3.3 and 4.1 below.)

[3] When I wrote [3], I was not sensitive to these issues; although I was working with classical logic, I used Beeson's version of the forcing clause for implication where the usual version would have worked just as well.

Axiomatic second-order arithmetic is often termed "analysis" because, by coding real numbers and continuous functions as sets of natural numbers, one can develop a workable theory of real analysis in this axiomatic framework. In fact, there is a long tradition of showing that one can get pretty far with restricted subsystems. Such research extends from the work of Weyl [81] and Hilbert and Bernays [42], through Takeuti [76], to contemporary work in the "reverse mathematics" program by Simpson, Friedman, and many others [72]. In the reverse mathematics tradition, one drops the schema of comprehension in favor of weaker set existence principles; and with theories that are too weak to prove Σ_1 comprehension one replaces the induction axiom by an induction schema for Σ_1 formulae with (number and set) parameters. Five theories have been singled out as representative of standard mathematical constructions:

- RCA_0: based on a Recursive Comprehension Axiom, i.e. comprehension for Δ^0_1 formulae with parameters
- WKL_0: based on a form of Weak König's Lemma, which asserts the existence of a path through any infinite tree on $\{0, 1\}$
- ACA_0: based on the Arithmetic Comprehension Axiom scheme
- ATR_0: based on Arithmetic Transfinite Recursion, i.e. arithmetic comprehension iterated along any well-ordering
- $\Pi^1_1\text{-}CA_0$: based on the Π^1_1 Comprehension Axiom scheme

The subscripted 0 indicates that one does not have the full schema of induction.

An *ω-model* of one of these theories is a model in which the first-order part is the standard structure of the natural numbers, and so amounts to a collection subsets of $\mathbb{N}$ with enough closure properties to satisfy the axioms. The smallest ω-model of ACA_0 is the collection of arithmetic sets, and the theory ACA_0 is a conservative extension of first order Peano arithmetic (PA), much the way that Gödel-Bernays-von Neumann set theory is a conservative extension of ZFC. By way of comparison, the smallest ω-model of RCA_0 is the collection of recursive sets. One can use this to obtain an interpretation of RCA_0 in $I\Sigma_1$, the fragment of Peano arithmetic in which induction is restricted to Σ_1 sentences. The idea is to represent the recursion-theoretic model internally, interpreting the second-order variables of RCA_0 by indices for recursive sets in $I\Sigma_1$. An old theorem, due to Parsons, Mints, and Takeuti independently, asserts that $I\Sigma_1$ is a conservative extension of primitive recursive arithmetic, for Π_2 sentences, in the following strong sense: if $I\Sigma_1$ proves $\forall x\ \exists y\ \varphi(x, y)$ for a Δ_0 formula $\varphi(x, y)$, then there is a function symbol f and a quantifier-free proof of $\varphi(x, f(x))$ in PRA. So, in sum, RCA_0 provides an axiomatic framework for recursive mathematics that is no stronger than primitive recursive arithmetic.

3.1 Weak König's Lemma

The theory WKL_0 augments RCA_0 with the following second-order axiom:

$$\forall T\ (T \text{ an infinite tree on } \{0, 1\} \rightarrow \exists P\ (P \text{ is a path through } T)).$$

Here a tree T on $\{0,1\}$ is defined to be a set of finite binary sequences closed under taking initial segments, and an infinite path through T is defined to be a set X such that every initial segment of the characteristic function of X lies in T. This principle, which essentially expresses the compactness of $\{0,1\}^\omega$ under the product topology, is interesting because it allows one to prove other compactness principles of mathematical interest; these include the Heine-Borel principle, and the compactness of first-order logic.

From Kleene's construction of an infinite recursive tree on $\{0,1\}$ with no recursive path, we know that the collection of recursive sets do not form an ω-model of WKL_0. Indeed, WKL_0 has no minimal ω-model (see [72]). Nonetheless, Harvey Friedman, who introduced the theory, was able to show that WKL_0, like RCA_0, is still Π_2-conservative over PRA.

The usual proof of König's lemma shows that every infinite tree T on $\{0,1\}$ has a path computable in T', the Turing jump of T. Kleene's counterexample shows that we cannot always find a path computable from T. But Jockusch and Soare [46] have shown that we can have the next best thing: every infinite binary tree T has a path P computable in T' that is furthermore low: the Turing jump of P, P', is also computable in T'. This very elegant result involves a simple construction in which one iteratively thins down the relevant infinite binary tree and extends a path through it. At stage n, one considers the nodes σ in the tree that provide enough information to show that for any P extending σ, $\varphi_n^P(0)$ halts. If the result of throwing away these nodes leaves an infinite binary tree, one does so, guaranteeing that at the end of the construction, $\varphi_n^P(0)$ does not halt. Otherwise, one does nothing, and $\varphi_n^P(0)$ is guaranteed to halt for every path P through the tree at hand. It is not hard to show, first, that this construction is recursive in T'; and second, that P' (which can be taken to be the set of indices n such that $\varphi_n^P(0)$ halts) is computable from T', since whether or not $\varphi_n^P(0)$ halts is determined by the construction. (See [21] for more detail.)

I have already noted, in the introduction, that such an argument can be viewed as a forcing construction. In the Jockusch-Soare proof one can take the conditions to be infinite binary trees, where a tree T_1 is stronger than another tree T_2 if $T_1 \subseteq T_2$. Harrington showed how to adapt this to the context of models of subsystems of arithmetic, so that given a model of RCA_0 and an infinite binary tree T in the sense of the model, one can force to add a path through T and close under relative recursion, to obtain another model of RCA_0 in which T has a path. The only real work is involved in showing that Σ_1 induction is preserved, and the proof that this is the case is based on the Jockusch-Soare idea. Iterating this process shows that any countable model of RCA_0 can be extended to a model of WKL_0, yielding the following:

Theorem 3.1 *WKL_0 is conservative over RCA_0 for Π^1_1 sentences.*

This is a strengthening of Friedman's theorem, since we have already seen that RCA_0 is conservative over primitive recursive arithmetic for Π_2 sentences.

Oddly enough, the model-theoretic argument does not tell us how to *translate* a proof of a Π^1_1 theorem in WKL_0 to one in RCA_0. Hájek [40] showed how to obtain an effective (and efficient) version of the conservation theorem, by constructing a recursion-theoretic model of WKL_0 in RCA_0 (in fact, in $I\Sigma_1$). This involved proving a stronger version of the low basis theorem, and then carefully carrying out an iterative construction in $I\Sigma_1$.

I achieved a similar result, independently, by formalizing the Harrington forcing argument in RCA_0, along the lines discussed in Section 2.4. Let σ range over finite sequences from $\{0, 1\}$, let σ_n denote the nth element of σ, and let T range over infinite binary trees, as above. To reason about a single generic path in RCA_0, define

$$T \Vdash t \in G \equiv \text{“}\{\sigma \mid \sigma_t = 0\} \text{ is finite”};$$

in other words, T forces $t \in G$ if all but finitely many nodes of T have value 1 at t. Unwinding definitions allows one to show that for any T and σ, T forces that σ is an initial segment of (the characteristic function of) G if and only if all but finitely many nodes of T are compatible with σ. In particular, T forces that every initial segment of G is an element of T, so:

Lemma 3.2 *RCA_0 proves that for every condition T, T forces “G is a path through T.”*

In order to show that Σ_1 induction is preserved, one first needs to see that forcing for a Σ_1 formula is again Σ_1:

Lemma 3.3 *For every Σ_1 formula $\exists x\ \theta(x, G)$, the assertion that $T \Vdash \theta(x, G)$ is equivalent to another Σ_1 formula, provably in RCA_0.*

One can show this by first noting that the formula $\exists x\ \theta(x, G)$ can be put in Kleene normal form, $\exists \sigma \subset G\ \theta'(\sigma)$, where θ' is Δ_0 and $\sigma \subset G$ means that σ is an initial segment of G; and that enough arithmetic is forced to demonstrate this equivalence. Unwinding definitions again shows that $T \Vdash \exists x\ \theta(x, G)$ is equivalent to the Σ_1 assertion that $\{\sigma \in T \mid \neg\theta'(\sigma)\}$ is finite.

Now it is not hard to see that RCA_0 proves that Σ_1 induction is forced: if a condition T forces $\varphi(0) \land \forall x\ (\varphi(x) \to \varphi(x+1))$ for some Σ_1 formula φ, then T forces $\varphi(0)$, and for every x, if T forces $\varphi(x)$, it also forces $\varphi(x+1)$. Applying Σ_1 induction in RCA_0 shows that for every x, T forces $\varphi(x)$, as required.

What all this shows is that in RCA_0 one can reason about a generic extension which satisfies all the axioms of RCA_0, except for perhaps recursive comprehension, and in which we have added a generic path through a single infinite binary tree. Obtaining the full result involves iterating the forcing internally, so that at each stage we add, generically, a new path through a tree and all the sets recursive in it. Care is required to ensure that the notion of a condition (essentially, a finite sequence of names for infinite binary trees) can be defined uniformly, but otherwise the argument is straightforward.

It is interesting to note that my approach and Hájek's really are different: Hájek's argument yields conservation principles for extensions of WKL_0 with certain collection principles, whereas mine seems to work better for even weaker theories. Some of these issues are discussed in [7].

3.2 Ramsey's theorem

Ramsey theory has long been a fruitful source of questions and methods in many branches of logic, including set theory, model theory, and recursion theory. The subject's use of infinitary and nonconstructive methods, often with explicit finitary consequences, makes it an interesting topic of study for proof theorists as well.

Let RT^n_k denote the infinitary version of Ramsey's theorem for k-colorings of n-elements sets. In other words, RT^n_k asserts that for every k-coloring of (unordered) n-tuples of natural numbers, there is an infinite homogeneous set, i.e. an infinite $S \subseteq \mathbb{N}$ such that every n-tuple from S gets the same color.

Recursion-theoretic interest in Ramsey's theorem seems to have originated with Specker, who, in 1966, showed that RT^2_2 fails in the recursive setting: there is a recursive coloring of pairs of natural numbers with no recursive infinite homogeneous subset. In 1970, Jockusch [45] presented a thorough analysis of the complexity of infinite homogeneous sets of recursive k-colorings. For example, a particular instance of one of his theorems shows that there is a recursive coloring of triples, such that $0'$, i.e. the halting problem, is computable from any infinite homogeneous set. Adapting the argument to the context of reverse mathematics shows that in fact RT^2_3 implies arithmetic comprehension over RCA_0. In fact, using these ideas, Simpson showed that, over RCA_0, ACA_0 is exactly equivalent to any of the statements RT^2_k, for $k \geq 3$.

What about RT^2_2, i.e. Ramsey's theorem for 2-colorings of pairs of natural numbers? Jockusch showed that a recursive 2-coloring of pairs such that no infinite homogeneous set is computable from $0'$. Since one can construct an ω-model of WKL_0 every set of which *is* computable from $0'$, this shows that WKL_0 does not prove RT^2_2.

It is natural to consider the converse: does RT^2_2 imply WKL_0, or even ACA_0, over RCA_0? Despite continued efforts from in the recursion theoretic community, it was not until the 1990's that Seetapun was able to show that RT^2_2 does not prove ACA_0. More generally, he showed that for all sets A and B, if A is not recursive in B and F is a 2-coloring which is recursive in B, then there is an infinite F-homogeneous set H such that A is not recursive in the join of B and H. In particular, if A is not recursive, there is a recursive 2-coloring of pairs such that A is not computable from any infinite homogeneous set. With a suitable iteration one can therefore construct an ω-model of RCA_0 which avoids $0'$, and such a model cannot satisfy ACA_0. Seetapun's construction is presented as a forcing argument in [70]. The question as to whether RT^2_2 implies WKL_0 over RCA_0 is still open, as is the problem of determining the first-order consequences of $RCA_0 + RT^2_2$.[4]

Recently Cholak, Jockusch, and Slaman [19] made some progress with respect to determining the strength of RT^2_2. For example, using a recursion-theoretic construction, they showed:

[4] Jeffry Hirst [43] has shown that $RCA_0 + RT^2_2$ implies Σ_2 collection; it is conceivable that the first-order consequences of $RCA_0 + RT^2_2$ are exactly those of $I\Sigma_1$ plus Σ_2 collection.

Theorem 3.4 *For every 2-coloring C of pairs of natural numbers, there is an infinite homogeneous set H that is low$_2$ in C, i.e. $H'' \leq_T C''$.*

Using a forcing analogue of the same methods, they obtained the following:

Theorem 3.5 *$RCA_0 + I\Sigma_2 + RT^2_2$ is conservative over $RCA_0 + I\Sigma_2$ for Π^1_1 sentences.*

As a forcing argument inspired by a recursion-theoretic construction, Theorem 3.5 bears a relationship to Theorem 3.4 that is analogous to the relationship between the Jockusch-Soare low-basis theorem and Harrington's conservation theorem for WKL_0. It seems likely that one can obtain an effective version of Theorem 3.5, providing an explicit interpretation of $RCA_0 + I\Sigma_2 + RT^2_2$ in $RCA_0 + I\Sigma_2$, along the lines discussed in 3.1. To date, however, this has not been carried out.

3.3 Other examples

Many constructions involving models of subsystems of second-order arithmetic are closely related to constructions in descriptive set theory (where the focus is, of course, on the standard model); see [68, 72]. For example, a forcing argument due to Steel can be used to show that the Σ^1_1 axiom of choice does not follow from Δ^1_1 comprehension.

Forcing has been used to shed light on other aspects of weak König's lemma. For example, Simpson and Smith [73] extended Harrington's argument to a Π_2 conservative extension of elementary arithmetic. Ferreira [32] obtained analogous results for a theory of polynomial-time computable arithmetic, and Fernandes [31] has recently extended this to obtain a conservation theorem for a principle of strict Π^1_1 reflection. Simpson, Tanaka, and Yamazaki [74] have shown that homogeneity properties of the Harrington forcing argument ensure that whenever WKL_0 proves $\forall X\ \exists! Y\ \varphi(X, Y)$, with φ arithmetic, then RCA_0 proves it as well. Brown and Simpson [15] have used Cohen forcing to show that a version of the Baire Category Theorem is also Π^1_1 conservative over RCA_0.

4 Intuitionistic theories

Just as classical forcing is useful in proving conservation results for classical theories, so too is intuitionistic forcing useful in proving conservation results for intuitionistic theories. It is also a surprisingly useful tool for interpreting classical theories in constructive ones. I will discuss one example of each type in Sections 4.1 and 4.2, and mention a few more examples in Section 4.3.

4.1 Goodman's theorem

First-order classical arithmetic, also known as Peano arithmetic (PA), has an intuitionistic counterpart known as Heyting arithmetic, or HA. In fact, Heyting arithmetic is obtained by taking any standard axiomatization of Peano arithmetic, but basing the theory on intuitionistic logic instead. Put the other way, Peano arithmetic can be fruitfully viewed as Heyting arithmetic together with the law of the excluded middle.

The set of finite types over the natural numbers is defined inductively, in the following way: $\mathbb{N}$ is a type, and if σ and τ are types, so is $\sigma \to \tau$. In the set-theoretic interpretation, of course, $\sigma \to \tau$ denotes the set of all functions from σ to τ. But, for restricted theories, more meager interpretations are available; for example, one can often interpret $\sigma \to \tau$ as a collection of suitably recursive functionals from σ to τ.

HA^ω is a finite-type conservative extension of Heyting arithmetic. One can view this as a many-sorted theory, with variables and quantifiers ranging over each of the finite types. In addition to the usual constants and functions on the natural numbers, one defines higher-type terms using explicit definition and a form of primitive recursion. Of course, HA^ω has a classical analogue, PA^ω, which is a conservative extension of PA.

What distinguishes PA^ω from full higher-order arithmetic (and HA^ω from higher-order intuitionistic arithmetic) is the absence of comprehension axioms. In a language based on function symbols, one typically identifies sets with their characteristic functions. In that case, the comprehension axioms have the form

$$\exists f^{\sigma \to \mathbb{N}}\ \forall x^\sigma\ (f(x) = 1 \leftrightarrow \varphi(x))$$

where φ is any formula in which f is not free. By avoiding such axioms, HA^ω and PA^ω provide a flexible language for formalizing portions of infinitary mathematics on the cheap, which is to say, without going beyond the axiomatic strength of first-order arithmetic.

In the language of finite type arithmetic, the axiom of choice, (AC), is the following schema:

$$\forall x^\sigma\ \exists y^\tau\ \psi(x, y) \to \exists f^{\sigma \to \tau}\ \forall x^\sigma\ \psi(x, f(x)).$$

Classically, this schema is very strong: applying it to the formula

$$(y = 1 \land \varphi(x)) \lor (y = 0 \land \neg\varphi(x))$$

yields the comprehension axiom for φ, so $PA^\omega + (AC)$ is as strong as full higher-order arithmetic. Intuitionistically, however, the schema is surprisingly weak. One can show, in HA^ω, that the axiom of choice is realized, under a standard realizability interpretation. Hence, whenever $HA^\omega + (AC)$ proves a formula φ, HA^ω proves that φ is realized. Furthermore, for negative formulae φ, i.e. formulae that do not involve $\lor$ or $\exists$, HA^ω can prove that being realized is equivalent to being true. This shows that $HA^\omega + (AC)$ is conservative over HA^ω for negative formulae. (See [13, 79] for the details.)

A theorem by Goodman provides a nicer result:

Theorem 4.1 *$HA^\omega + (AC)$ is a conservative extension of HA^ω, and hence HA, for arithmetic sentences.*

Goodman gave two proofs of this result, and Beeson has found it useful to present the second of these as the composition of a forcing argument with the realizability argument above. The realizability argument still works if one uses realizers that are computable relative to a numeric function f. Fixing an arbitrary arithmetic formula φ, the trick is to force to add a generic function f so that φ is realized, relative to f, if and only if φ is true. This function f need only code appropriate witnesses to true subformulae of φ that are of the form $\exists x\ \psi$, and, similarly, choose an appropriate disjunct for true subformulae of the form $\psi \vee \theta$. Finding the appropriate forcing notion is easy and straightforward. Beeson notes that, using a similar argument, Goodman's theorem can be extended to include the extensionality axiom in the source theory; see [13].

4.2 Interpreting classical theories in constructive ones

The original goal of Hilbert's proof-theoretic program was to provide finitary consistency proofs for classical mathematics. After Gödel's incompleteness theorem showed this goal to be unattainable, the emphasis shifted to a modified version of Hilbert's program, wherein the more general goal is to justify classical theories relative to *constructive* ones.

Perhaps the most compelling reduction of this kind involves a direct interpretation of the classical theory in its constructive counterpart. In that respect, the double-negation translation discussed in Section 2.1 provides a remarkably effective tool. In particular, it reduces *PA* to *HA*, and works just as well for higher-order and higher-type versions of arithmetic as well. With some additional work, it can even be used to interpret Zermelo-Fraenkel set theory in a suitable intuitionistic version, like Friedman's *IZF* (see [13]).

But the double-negation translation does not always work. Remember, the theorem tells us that if φ is provable classically from a set of axioms Γ, then φ^N is provable intuitionistically from Γ^N. The net result is therefore only interesting insofar as one can make constructive sense of the doubly-negated axioms, Γ^N. Fortunately, the double-negation translation of an induction axiom is again an induction axiom, yielding the reduction of *PA* to *HA*. But, for example, the double-negation interpretation of Σ_1 induction involves induction on predicates of the form $\neg\neg\exists x\ A(x,y)$ (or equivalently, $\neg\forall x\ \neg A(x,y)$); so $I\Sigma_1$ is not immediately interpreted in its intuitionistic counterpart, $I\Sigma_1^i$. For another example, the double negation translation of the Σ^1_1 axiom of choice is of the form

$$\forall x\ \neg\neg\exists Y\ \varphi(x,Y) \rightarrow \neg\neg\exists Y\ \forall x\ \varphi(x,Y_x)$$

where φ is arithmetic. Whereas the conclusion of this implication is, intuitionistically, a weakening of the conclusion Σ^1_1 choice axiom, so is the hypothesis. So, intuitionistically, the Σ^1_1 axiom of choice does not imply its double negation.

It turns out that we can use the latitude in defining "$p \Vdash \bot$" to repair the double-negation translation in cases like these. An early instance of this idea can be found in Buchholz' interpretation of theories of inductive definitions $ID_{<\alpha}$ in their intuitionistic counterparts $ID^i_{<\alpha}$, in [17]. More recently, Coquand realized that this idea could be used to interpret $I\Sigma_1$ in $I\Sigma_1^i$. He and Hofmann [27] then extended the interpretation to Buss' theory S_2^1 of bounded arithmetic. Independently, in [4], I extended the interpretation to bounded arithmetic, as well as to subsystems of second-order arithmetic based on Σ_1^1 choice and various fragments of admissible set theory.

Let us consider Σ_1 induction, for example. To repair the double-negation translation, it suffices to add Markov's principle:

$$\neg \forall x\ A \rightarrow \exists x\ \neg A,$$

where A is any Δ_0 (or primitive recursive) formula; with this principle, a double-negated Σ_1 sentence becomes equivalent to one that is Σ_1. We will be done if we can in turn use a forcing relation to interpret intuitionistic $I\Sigma_1^i$ plus Markov's principle in $I\Sigma_1^i$ alone.

To do so, take conditions p to be (codes for) finite sets of Π_1 sentences,

$$\{\forall x\ A_1(x), \forall x\ A_2(x), \ldots, \forall x\ A_k(x)\}.$$

Define $p \leq q$ to be $p \supseteq q$. For θ atomic, define $p \Vdash \theta$ to be

$$\exists y\ (A_1(y) \wedge \ldots \wedge A_k(y) \rightarrow \theta).$$

In particular, since $I\Sigma_1$ proves the law of the excluded middle for Δ_0 (or primitive recursive) formulae, $p \Vdash \bot$ is equivalent to

$$\exists y\ (\neg A_1(y) \vee \ldots \vee \neg A_k(y)).$$

This is nothing more than an intuitionistically strong negation to the conjunction of the formulae in p, i.e. one asserts the existence of a particular counterexample.

Lemma 4.2 *The following are provable in $I\Sigma_1^i$:*

1. $\{\forall x\ A(x)\} \Vdash \forall x\ A(x)$
2. *If* $p \Vdash \neg\forall x\ A(x)$*, then* $p \Vdash \exists x\ \neg A(x)$.
3. $\Vdash \neg\forall x\ A(x) \rightarrow \exists x\ \neg A(x)$

For the first statement, we have

$$\begin{aligned} \forall x\ A(x) \Vdash \forall x\ A(x) &\equiv \forall z\ (\forall x\ A(x) \Vdash A(z)) \\ &\equiv \forall z\ \exists y\ (A(y) \rightarrow A(z)) \end{aligned}$$

which is easily verified, taking y to be z. For the second statement, let p be the set $\{\forall x\ B_1(x), \ldots, \forall x\ B_k(x)\}$, and suppose $p \Vdash \neg\forall x\ A(x)$. Then whenever $q \Vdash \forall x\ A(x)$, we have $p \cup q \Vdash \bot$. By 1, we have $p \cup \{\forall x\ A(x)\} \Vdash \bot$, i.e.

$$\exists y\ (B_1(y) \wedge \ldots \wedge B_k(y) \wedge A(y) \rightarrow \bot).$$

This implies

$$\exists x, y\ (B_1(y) \wedge \ldots \wedge B_k(y) \rightarrow \neg A(x)),$$

which is equivalent to

$$\exists x\ (p \Vdash \neg A(x)).$$

But this is just $p \Vdash \exists x\ \neg A(x)$, as required. The third statement follows immediately from the second, by the definition of forcing for an implication. But this is just the statement that Markov's principle is forced.

This application of forcing may seem strange, since there is no generic object being added. There are no names; the domain is constant at each condition, and is just the domain of the ground model. (Something of a model-theoretic interpretation can be found in [8].) The forcing conditions only serve to make forcing a negation, or forcing falsity, carry additional constructive information. The fact that the argument works only highlights the flexible and surprising nature of the forcing relation.

4.3 Other applications

Other applications of forcing in an intuitionistic setting are discussed in Beeson [13]. In particular, Beeson [11] shows that whenever a suitable constructive theory proves the totality of a computable functional from $2^{\mathbb{N}}$ to $\mathbb{N}$, there is a natural number k such that the theory proves that the functional is in fact bounded by k. Similarly, Hayashi [41] shows that suitable constructive theories are closed under a bar-induction rule. Lubarsky [54] uses forcing construtions to obtain independence results for intuitionistic Kripke-Platek set theory. General forcing frameworks for intuitionistic set theory are discussed in [12] and [39]; the latter discusses connections to sheaf models and Grothendieck coverings. For further relationships between intuitionistic logic and forcing see [33].

There are a number of sheaf constructions in categorical logic that are not presented in terms of axiomatic theories, but can be perhaps turned into conservation theorems when presented in more syntactic terms. Moerdijk and Reyes' constructions [60] of models of smooth infinitesimal analysis provide examples; see also the discussion of intuitionistic models of nonstandard analysis at the end of Section 5.2.

5 Point-free model theory

A central theme in the modern theory of sheaves is that the study of sheaves over a topological space X can be cast in terms of its lattice of open sets $\mathcal{O}(X)$, without reference to the points of X; and that this approach carries over to more general "point-free" spaces. This idea, i.e. replacing the points, or maximal elements, of a space by suitable systems of approximations, has found relevance in constructive mathematics. For example, real numbers can be approximated by intervals with rational endpoints, a maximal ideal can be approximated by subideals, and so on.

I have neither the space nor the ability to provide an adequate overview of constructive, point-free approaches to mathematics. For that, I will refer the reader to [48, 47, 34, 35, 22, 69]; for examples of the use of point-free thinking in extracting constructive proofs from classical arguments, see [23, 24, 26, 29]. My goal in this section is, rather, to describe some applications of "point-free" ideas to constructivizing model-theoretic arguments. Many model-theoretic constructions are based on either the compactness or completeness theorem for first-order logic, which, in turn, amounts to having an appropriate "maximal" object: a maximally consistent (or maximally satisfiable) set of sentences, or, equivalently, a maximal filter in an associated Boolean algebra. Often it is irrelevant which *particular* maximal object is used, and, in such cases, one can often constructivize the argument by reasoning about such maximal objects generically. In the example above, it may be sufficient to work with finite sets of sentences, with the knowledge that if S is a finite consistent set and φ is any sentence, S can be consistently extended by adding φ or $\neg\varphi$. For another example, most ultrapower constructions work for *any* maximal filter extending the Fréchet filter; it is often sufficient to reason about what is forced to be true by arbitrary filters, keeping in mind that if F is a filter and A is any set, either A or its complement can be added to F. The net result is that classical model-theoretic constructions can often be recast as classical forcing constructions; and often these can, in turn, be internalized as syntactic interpretations.

5.1 Constructive cut elimination theorems

Gentzen's cut-elimination theorem for first-order logic says that any proof in a suitable sequent calculus (for classical, intuitionistic, or minimal logic) can be transformed into one that is cut-free. The cut rule is essentially a form of modus ponens, so the cut elimination theorem asserts, roughly, if one can prove a theorem ψ by proving a lemma $\varphi \to \psi$ and then proving φ, then one can prove ψ directly. This avoidance of detours makes it possible to read off useful information from cut-free proofs.

The natural way to prove a cut-elimination theorem is, of course, to provide an explicit procedure for transforming any proof with cuts to one without. But there is an equally straightforward, if less direct, model-theoretic way to establish such a theorem: show that the relevant system *with* cut is sound with respect to a given semantics, whereas the system *without* cut is complete. Taken together these imply that anything provable with cut is valid, and hence provable without.

Indeed, cut elimination for systems of higher-order logic is often proved this way, using standard (many-sorted) Henkin semantics. See, for example, Hayashi's proof of cut elimination for simple type theory, discussed in [77]. But there is a long tradition of using algebraic semantics to prove cut elimination, often described in terms of a Kripke model or forcing relation; see, for example [16, 30, 63]. The resulting proofs lie somewhere between the model-theoretic arguments and the explicitly syntactic ones: they typically have more algorithmic content than the former, but are less dependent on the details of a particular proof system than the latter. As a result, algebraic proofs provide a nice compromise between the two.

In [5], I show how one can interpret a standard model-theoretic proof of cut-elimination for classical first-order logic in terms of a forcing relation, and from that extract an explicit algorithm for eliminating cuts. There I also consider a related constructive proof of cut-elimination for intuitionistic first-order logic, which is based on a proof by Buchholz [16]; a novel variant of the double-negation translation, introduced in [5], reduces the classical cut-elimination theorem to the intuitionistic one. In the latter proof, conditions $p, q, r, \ldots$ are taken to be finite sets of formulas, and p is said to be stronger than q if $p \supseteq q$. If A is atomic, $p \Vdash A$ is defined to mean that there is a cut-free proof of the sequent $p \Rightarrow A$. Forcing is then extended to all formulas in the language in the usual way, modulo an appropriate covering relation (see the discussion in Section 2.3). A straightforward induction on φ then allows one to show:

Lemma 5.1 *For any formula φ,*

1. *$\{\varphi\} \Vdash \varphi$, and*
2. *if $p \Vdash \varphi$, there is a cut-free proof of $p \Rightarrow \varphi$.*

In particular, if φ is provable in intuitionistic logic, then, by soundness, it is forced; and then the second clause implies it has a cut-free proof. The argument extends to higher-order logic as well; see Buchholz [16].

Similar ideas can be used to obtain algebraic proofs of other results that can be obtained by both model-theoretic and proof-theoretic methods. See, for example Coquand's treatment of Herbrand's theorem and Skolem functions in [25], or the uniform method of obtaining a number of conservation results in [8]. Other proof-theoretic applications of forcing ideas are sketched in [14].

5.2 Weak theories of nonstandard arithmetic

The subject of nonstandard analysis has both a semantic and a syntactic side. As practiced by Abraham Robinson, nonstandard analysis is a model-theoretic technique: one proves theorems in an appropriately saturated elementary extension of some suitable universe (e.g. second- or higher-order arithmetic, or a universe of sets), and then "transfers" results back to the original, standard structure. Kreisel [51] and then Nelson [62] showed that one can treat this process axiomatically; for example, Nelson's *Internal Set Theory* is a conservative extension of *ZFC* with a predicate for the "standard" sets and axioms characterizing them relative to a saturated nonstandard extension.

Model-theoretic constructions of such nonstandard universes typically use either compactness or an ultrapower. It is therefore perhaps surprising that one can obtain conservation results for nonstandard theories that are quite weak. In the 1960's, Harvey Friedman showed that a nonstandard version of Peano arithmetic with standard induction and transfer principles is conservative over Peano arithmetic (see [66] for a formulation and a proof different from Friedman's). More recently, Suppes, Chauqui, and Sommer [20, 75] have considered weak nonstandard theories, whose consistency can be proved in (fragments of) primitive recursive arithmetic.

In [2], I sharpen and extend many of these results by providing a uniform method of turning theories of bounded arithmetic (like primitive recursive arithmetic, elementary arithmetic, and polynomial-time computable function arithmetic) into nonstandard versions, and, furthermore, interpreting the nonstandard versions into standard ones. Consider, for example, primitive recursive arithmetic, *PRA*. One obtains a nonstandard version, *NPRA*, by adding to the language of *PRA* a predicate $st(x)$ ("x is standard") and a constant, ω, intended to denote a nonstandard number. Let *NPRA* consist of *PRA* plus the following axioms:

- $\neg st(\omega)$
- $st(x) \wedge y < x \rightarrow st(y)$
- $st(x_1) \wedge \ldots \wedge st(x_k) \rightarrow st(f(x_1, \ldots, x_k))$, for each function symbol f
- $\forall$-transfer without parameters: $\forall^{st} \vec{x}\ \psi(\vec{x}) \rightarrow \forall \vec{x}\ \psi(\vec{x})$, for ψ quantifier-free and standard with the free variables shown.

The first three assert that there is a nonstandard element, ω, and that the standard numbers are closed downwards and under functions in the language. The last is a very restricted form of the usual transfer schema, which asserts that the entire system of numbers form an elementary extension of the standard part. A fairly straightforward application of compactness shows:

Theorem 5.2 *Suppose NPRA proves $\forall^{st} x\ \exists y\ \varphi(x, y)$, with φ quantifier-free in the language of PRA. Then PRA proves $\forall x\ \exists y\ \varphi(x, y)$.*

In particular, the conclusion holds if *NPRA* proves either $\forall x\ \exists y\ \varphi(x, y)$ or $\forall^{st} x\ \exists^{st} y\ \varphi(x, y)$.

This result extends to higher-type theories as described in Section 4.1, making it possible to develop theories of analysis and measure theory smoothly. In [2], I show that, furthermore, it is possible to obtain the conservation result by a direct interpretation of the higher-type nonstandard theories into the corresponding standard ones. The interpretation uses a forcing relation, which, in a sense, internalizes the compactness construction. Elements of the nonstandard universe are named by sequences dependent on ω. One can take conditions to be pairs $\langle \alpha, f \rangle$, where α and f are, respectively, a predicate and a function on the natural numbers, satisfying

$$\forall z\ \exists \omega\ (f(\omega) > z \wedge \alpha(\omega)).$$

Intuitively, this says that there are interpretations of ω satisfying α and making f arbitrarily large. Two numbers, represented by sequences $x(\omega)$ and $y(\omega)$, respectively, are forced to be equal if

$$\exists z\ \forall \omega\ ((f(\omega) > z \wedge \alpha(\omega)) \rightarrow x(\omega) = y(\omega)).$$

In other words, they are forced to be equal if and only if they are equal for all ω satisfying α for which $f(\omega)$ is large enough. Forcing for other atomic formulae is defined similarly. In particular, a sequence $x(\omega)$ is forced to be standard by $\langle \alpha, f \rangle$ if there is a uniform bound on $x(\omega)$ for all ω for which $f(\omega)$ is large enough.

There is a strong intuition that nonstandard arguments can be translated to standard ones by replacing "nonstandard" everywhere by "large enough." The translation above is, however, the closest thing I know of to a justification of this intuition. In many ways, the translation is natural; for example, nonstandard intervals of natural numbers translate to intervals that are arbitrarily large, in an appropriate sense. But other features are somewhat novel. For example, real numbers are named by bounded sequences of rationals; by the Bolzano-Weierstrass theorem every such sequence will have a convergent subsequence, but the forcing translation provides a way of reasoning about such reals generically.

To be sure, without transfer principles and induction on the standard elements, these theories are fairly weak. But the interpretations are flexible enough to allow one to add such principles, provided that one augments the standard target theories accordingly. As a result, these theories provide a good framework for studying the methods of nonstandard analysis to determine which principles are needed for carrying out classical mathematical arguments, and how they are used.

There has recently also been a good deal of interest in *intuitionistic* theories of nonstandard arithmetic and analysis, supported by the observation that many nonstandard arguments are essentially constructive, modulo, of course, the use of nonstandardness. See, for example, [59, 9]. The constructions in [59] are inspired by more general sheaf-theoretic constructions of models of not only nonstandard analysis, but synthetic differential geometry: see, for example, [58, 60, 65].

5.3 Eliminating Skolem functions

A *Skolem axiom* has the form

$$\forall \vec{x}, y \; (\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, f(\vec{x}))),$$

where f is a new function symbol introduced to denote a "Skolem function" for φ. Intuitively, f picks out witnesses y to $\varphi(\vec{x}, y)$ whenever possible, so that if anything satisfies $\exists y \; \varphi(\vec{x}, y)$, $f(\vec{x})$ does.

An easy model-theoretic argument shows that such axioms can be added conservatively to any first-order theory. Suppose T does not prove a sentence ψ which does not mention the function f above; then $T \cup \{\neg\psi\}$ is consistent, and hence has a model (even a countable one, assuming the language of T is countable). Expanding this model by any Skolem function for φ shows that T together with the Skolem axiom does not prove ψ either.

This is one of the few cases I know of in logic where an explicit syntactic argument is considerably more difficult. Hilbert and Bernays first presented such a proof in [42], using the epsilon substitution method. Maehara later presented a proof using cut-elimination (his proof is discussed in [77]). Another proof, due to Shoenfield, is found in [71]. All these procedures allow, in the worst case, an iterated exponential increase in the length of proof; we do not know whether it is possible to do better, nor do we have nontrivial lower bounds on the increase in length.

The forcing interpretations I have described so far, however, can be carried out in polynomial time, and, in particular, lead to at most a polynomial increase in proof length. This suggest a way of obtaining an efficient elimination of Skolem functions in the context of an axiomatic theory: force to add the requisite functions, and describe the resulting model in the language of the underlying theory. To interpret the single Skolem axiom for φ above, take conditions to be finite partial functions p satisfying

$$\forall \vec{x} \in dom(p)\ \forall y\ (\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, p(\vec{x}))).$$

In other words, insofar as p is defined, p looks like a Skolem function for φ. By genericity, to interpret the Skolem axiom, it suffices to show that if p is any condition and $\vec{x}$ is any sequence of values, p can be extended to a condition q that is defined at $\vec{x}$. This requirement can clearly be met, by setting $q(\vec{x})$ to any y satisfying $\varphi(\vec{x}, y)$, if there is one, and 0 otherwise. Iterating this procedure carefully allows one to handle arbitrary sequences of (possibly nested) Skolem axioms.

For the argument to go through, one only need to know that the target theory has enough strength to code finite partial functions. This is a very meager requirement, and, since partial functions can be represented as sequences of ordered pairs, any "sequential" theory of arithmetic suffices. Modulo the details (carried out in [6]), we then have the following partial answer to Pudlák's question:

Theorem 5.3 *One can eliminate Skolem axioms in polynomial time from any theory in which one has a suitable coding of finite partial functions.*

6 Conclusions

Metamathematical research in proof theory is predicated on the assumption that reflection on the language, concepts, and methods of mathematics, and the representation of those methods in syntactic terms, is worthwhile from mathematical, philosophical, and computational points of view. Here I have tried to show how forcing touches on themes that are central to the subject. By offering a means of reasoning about complex generic objects in terms of approximations to them, forcing provides a number of ways of interpreting abstract or infinitary principles, in constructive or otherwise explicit terms.

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