# Mathematics and Language

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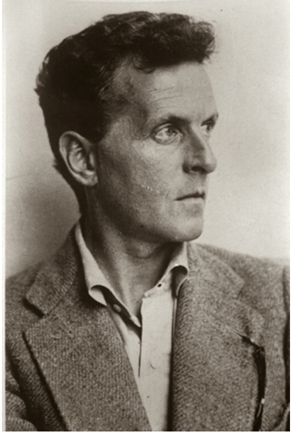
I learned empirically that this came out this time, that it usually does come out; but does the proposition of mathematics say that? . . . The mathematical proposition has the dignity of a rule.

*So* much is true when it's said that mathematics is logic: its moves are from rules of our language to other rules of our language. And this gives it its peculiar solidity, its unassailable position, set apart.

— Ludwig Wittgenstein



. . . it seemed to me one of the most important tasks of philosophers to investigate the various possible language forms and discover their characteristic properties. While working on problems of this kind, I gradually realized that such an investigation, if it is to go beyond common-sense generalities and to aim at more exact results, must be applied to artificially constructed symbolic languages.. . . Only after a thorough investigation of the various language forms has been carried through, can a well-founded choice of one of these languages be made, be it as the total language of science or as a partial language for specific purposes.

— Rudolf Carnap

Physical objects, small and large, are not the only posits.... the abstract entities which are the substance of mathematics... are another posit in the same spirit. Epistemologically these are myths on the same footing with physical objects and gods, neither better nor worse except for differences in the degree to which they expedite our dealings with sense experiences.

— W. V. O. Quine

"When I use a word," Humpty Dumpty said in rather a scornful tone, "it means just what I choose it to mean — neither more nor less."

"The question is," said Alice, "whether you can make words mean so many different things."

"The question is," said Humpty Dumpty, "which is to be master — that's all."

— Lewis Carroll

# Philosophical puzzles

- Mathematics tells us about the world, but not vice-versa.
- Mathematical objects are not located in space or time.
- Mathematics delivers (near?) certainty.

# Lowbrow answers

- Mathematics is governed by mathematical norms.
- We learn these norms from parents, teachers, ...
- We come to have mathematical knowledge by following these norms correctly.

But why are the norms the way they are, and why do they tell us anything about the world?

# The linguistic turn

Mathematics is part of our language.

- Linguistic norms govern the way we describe the world.
- We have adopted these norms because they are useful.

These themes (with variations) occur throughout Wittgenstein, Carnap and the Logical Positivists, and Quine.

Only empirical explanation is possible for why we have come to accept the basic principles that we do and why we apply them as we do—for why we have mathematics and why it is at it is. But it is only within the framework of mathematics as determined by this practice that we can speak of mathematical necessity. In this sense, which I believe Wittgenstein was first to fully grasp, mathematical necessity rides on the back of empirical contingency.

— William Tait

# The function concept

Today, we have the notion of a function $f : A \to B$ between any two "sets" or "domains."

From the time of Euler (ca. 1750) through the nineteenth century, mathematicians used the word exclusively for functions on the real or complex numbers.

They studied

- sequences and series
- number-theoretic functions ("symbols", "characters")
- geometric transformations ("affinities," "colineations")
- permutations ("substitutions")

and more, but these were not subsumed under a general concept.

## The function concept

Nineteenth century methodological changes:

1. Unification / generalization of the function concept
2. Liberalization of the function concept
3. Extensionalization of the function concept
4. Reification of the function concept

Rebecca Morris and I focused on an illuminating case study.

# Dirichlet's theorem

Theorem

*Let $m$ and $k$ be relatively prime. Then the arithmetic progression $m, m+k, m+2k, \ldots$ contains infinitely many primes.*

For example, there are no primes in the sequence

$$6, 15, 24, 33, 42, 51, \ldots.$$

There are infinitely many primes in the sequence

$$5, 14, 23, 32, 41, 50, \ldots$$

# Dirichlet's theorem

Legendre assumed this in 1798, in giving a purported proof of the law of quadratic reciprocity.

Gauss pointed out this gap, and presented two proofs of quadratic reciprocity in his *Disquisitiones Arithmeticae* of 1801.

He ultimately published six proofs of quadratic reciprocity, and left two more proofs in his *Nachlass*. But he never proved the theorem on primes in an arithmetic progression.

Dirichlet's 1837 proof is notable for the sophisticated use of analytic and algebraic methods to prove a number-theoretic statement.

# Dirichlet's theorem

Modern presentations of Dirichlet's proof rely on the notion of a *Dirichlet character*.

Consider the sequence

$$3, 13, 23, 33, \ldots$$

The common difference is 10.

The numbers relatively prime to 10, $\{1, 3, 7, 9\}$ form a group, with multiplication modulo 10.

A *character* on this group is a homomorphism to the complex numbers, for example

$$\chi(1) = 1, \quad \chi(3) = i, \quad \chi(7) = -i, \quad \chi(9) = -1.$$

# Dirichlet's theorem

Characters are treated as ordinary mathematical objects like the natural numbers.

1. The characters modulo $k$ form a group.
2. We define functions $L(s, \chi)$, called *Dirichlet L-series*, whose second argument is a character.
3. We write $\sum_{\chi} \bar{\chi}(m) L(s, \chi)$.

None of these features are present in Dirichlet's proof.

# Dirichlet's 1837 proof

In fact, there is no notion of character there at all!

Dirichlet wrote expressions

$$\theta^{\alpha}\varphi^{\beta}\omega^{\gamma}\omega'^{\gamma'}\dots,$$

where we would write $\chi(n)$.

Where we would write

$$\sum_{\chi\in\widehat{(\mathbb{Z}/k\mathbb{Z})^*}} \overline{\chi(m)}\log L(s,\chi),$$

Dirichlet wrote

$$\frac{1}{K}\sum \Theta^{-\alpha_m\mathfrak{a}}\,\Phi^{-\beta_m\mathfrak{b}}\Omega^{-\gamma_m\mathfrak{c}}\Omega^{-\gamma'_m\mathfrak{c}'}\cdots\log L_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{c}',\dots}.$$

# A timeline

Morris and I studied a history of presentations:

- Dirichlet 1837: Dirichlet's original proof
- Dirichlet 1840, 1841: extensions to Gaussian integers, quadratic forms
- Dedekind 1863: presention of Dirichlet's theorem
- Dedekind 1879, Weber 1882: characters on arbitrary abelian groups
- Hadamard 1896: presentation of Dirichlet's theorem and extensions
- de la Vallée Poussin 1897: presentation of Dirichlet's theorem and extensions
- Kronecker (1901, really 1870's and 1880's): constructive, quantitative treatment
- Landau 1909, 1927: presentation of Dirichlet's theorem and extensions

# Summary

Over time:

- The notion of a character was defined.
- Authors isolated general properties of characters.
- Authors got used to summing over characters, rather than representing data.
- Authors got used to functional dependences on characters, rather than representing data.
- Authors began to adopt extensional characterizations and classifications of characters.
- The use of explicit symbolic representations for the characters diminished and was ultimately eliminated.

# Methodological benefits

Turning characters into “things” brought a number of benefits:

- expressions were simplified
- proofs became more modular
- proofs became easier to understand and check
- key notations and relationships were made salient
- concepts were reusable and generalizable

On the one hand, this is a metaphysical shift.

On the other, it is just “language engineering.”

The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules. The most comprehensive formal systems that have been set up hitherto are the system of *Principia mathematica (PM)* on the one hand and the Zermelo-Fraenkel axiom system of set theory (further developed by J. von Neumann) on the other. These two systems are so comprehensive that in them all methods of proof used today in mathematics are formalized, that is, reduced to a few axioms and rules of inference. One might therefore conjecture that these axioms and rules of inference are sufficient to decide *any* mathematical question that can at all be formally expressed in these systems. It will be shown below that this is not the case...

# Modularity and complexity

In computer hardware: from logic gates, to bits and complex operations, to processors and memory, to complex systems, . . .

In computer software: from simple operations, to library operations, to subroutines, to complex programs, . . .

In formal mathematics: from basic logical operations, to definitions and properties, to more complex theorems and proofs, to theories. . .

Contemporary "proof assistants" now make it possible to construct complex formal proofs.

# Formal Proofs

The prime number theorem:

```
theorem PrimeNumberTheorem: "(λn. pi n * ln n / n) ⟶ 1"
```

where

```
pi n ≡ card {p. p ≤ n ∧ p ∈ prime}
```

The Feit-Thompson (Odd Order) Theorem:

```
Theorem Feit_Thompson (gT : finGroupType) (G : {group gT}) :
  odd #|G| → solvable G.

Theorem simple_odd_group_prime (gT : finGroupType)
    (G : {group gT}) :
  odd #|G| → simple G → prime #|G|.
```

# Formal Proofs

The Kepler conjecture (Hales' theorem):

```
∀V. packing V => (∃c. ∀r. &1 <= r =>
  &(CARD(V INTER ball(vec 0,r))) <=
    pi * r pow 3 / sqrt(&18) + c * r pow 2))
```

The Blakers-Massey theorem:

```
blakers-massey : ∀ {x₀} {y₀} (r : left x₀ ≡ right y₀) →
  is-connected (n +2+ m) (hfiber glue r)
```

# Formal Proofs

A definition of the natural numbers:

```
inductive nat : Type :=
| zero : nat
| succ : nat → nat
```

A definition of addition:

```
definition add : nat → nat → nat
| add x zero := x,
| add x (succ y) := succ (add x y)
```

Some notation:

```
notation ‘ℕ‘ := nat
notation 0 := zero
notation x ‘+‘ y := add x y
```

# Formal Proofs

Proving the commutativity of gcd:

```
theorem gcd.comm (m n : ℕ) : gcd m n = gcd n m :=
dvd.antisymm
  (dvd_gcd !gcd_dvd_right !gcd_dvd_left)
  (dvd_gcd !gcd_dvd_right !gcd_dvd_left)

theorem gcd_comm: "gcd (m::nat) n = gcd n m"
by (auto intro!: dvd.antisym)

Lemma gcdnC : commutative gcdn.
Proof.
move=> m n; wlog lt_nm: m n / n < m.
  by case: (ltngtP n m) => [||-> //]; last symmetry; auto.
by rewrite gcdnE -{1}(ltn_predK lt_nm) modn_small.
Qed.
```

# Formal proofs and formal languages

The point: whenever someone communicates with a computational proof assistant, they are speaking a formal mathematical language.

This language has been designed for a specific purpose, namely, to enable users to develop formal theories smoothly and efficiently.

# Taking stock

We have discussed:

- Informal mathematical language, and its evolution in the 19th century.
- Formal mathematical language, and its use in proof assistants.

In common:

- Both are uses of language convey mathematical content and support mathematical reasoning.
- In both cases, the languages we use can either serve our purposes well, or not.

# Differences

Differences between informal and formal mathematical languages:

- centuries vs. decades
- entire mathematical community vs. a design team
- implicit vs. explicit design decisions
- existence / nonexistence of a "reference manual"

# The central thesis

Thesis: formal mathematical languages like the ones used by proof assistants provide informative *models* of informal mathematical language.

Conversely, understanding informal mathematical language will help us design better theorem provers.

# Benefits

Understanding formal mathematical languages as models of informal languages can help us understand:

- The structure of mathematics (and the importance of modularity).
- The nature of mathematical reasoning and problem solving.
- The stability and reinterpretability of mathematical language over time.

# Conclusions

Let us get over the worry that there is something inherently mysterious or dubious about mathematical objects, and start paying attention to the things that really matter.

We need a science that can help us understand the considerations that bear upon our choices of mathematical norms, and the philosophy of mathematics should be that science.