The Role of the Diagram in Euclid's *Elements*

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(including joint work with Ed Dean, John Mumma, and Ben Northrop)

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Sequence of lectures

1. Mathematical Understanding
2. The History of Dirichlet's Theorem
3. Formalization and Interactive Theorem Proving
4. The Role of the Diagram in Euclid's *Elements*
5. Modularity in Mathematics

Outline

I will start with some thoughts on the relationship between cognitive science and the philosophy of mathematics.

Then:

1. Euclidean diagrammatic reasoning
2. The formal system, E
3. Soundness and completeness
4. Implementations
5. Conclusions

Logic and psychology

Edmund Husserl's *Philosophy of Arithmetic* of 1891 aimed,

> *. . . through patient investigation of details, to seek foundations, and to test noteworthy theories through painstaking criticism, separating the correct from the erroneous, in order, thus informed, to set in their place new ones which are, if possible, more adequately secured.*

He cast the work as a sequence of "psychological and logical investigations," providing a psychological analysis

> *. . . of the concepts multiplicity, unity, and number, insofar as they are given to us authentically and not through indirect symbolizations.*

Logic and psychology

Husserl had been influenced by Wilhelm Wundt, the "founder of experimental psychology," who aimed to

- make psychology scientific, and
- study inner life through "introspection."

Wundt's *Logik*:

- principles of reasoning employed in the sciences have their origins in psychological processes;
- these principles are justified by the fundamental role they play in thought.

This points to a unification of philosophy and psychology.

Logic and psychology

Husserl's *Philosophy of Arithmetic* was just that: a study of the way concepts arise in thought, and the role they play.

Concepts are described in dynamic terms, vis-à-vis mental operations:

- "noticing," "focusing attention"
- "ignoring," "disregarding"
- "seeing . . . as"

Concepts so analyzed: "something," "unit," "one", "collective combination," "multiplicity," "number."

Logic and psychology

For example, a "collective combination" involves *seeing* multiple objects individually and *as* a totality:

a cup, and pen, and a piece of chalk

One obtains a "multiplicity" by *disregarding* the particular nature of the elements:

a something, and a something, and a something

One obtains a "number" by *thinking of* a multiplicity as an answer to the question, "how many?".

Logic and psychology

"To disregard or abstract from something means merely to give it no special notice. The satisfaction of the requirement wholly to abstract from the peculiarities of the contents thus absolutely does not have the effect of making those contents, and therewith their combination, disappear from our consciousness. The grasp of the contents, and the collection of them, is of course the precondition of the abstraction. But in that abstraction the isolating interest is not directed upon the contents, but rather exclusively upon their linkage in thought – and that linkage is all that is intended."

Logic and psychology

Frege's review:

> *"We attend less to a property and it disappears. By making one characteristic after another disappear, we get more and more abstract concepts. . . Inattention is a most efficacious logical faculty; presumably this accounts for the absentmindedness of professors."*

From there:

- Husserl and continental philosophy: transcendental idealism
- Frege, Russell, Wittgenstein, Quine: study of linguistic practices

Logic and psychology

Is it time to reconsider?

- Calls for a "new epistemology" of mathematics, or a "philosophy of real mathematical practice."
- Advances in cognitive science, identifying "core" systems of cognition.

Two reactions:

- Optimistic: embrace the role of psychology in the philosophy of mathematics.
- Cautious: distinguish "philosophically interesting" from "merely cognitive."

Logic and psychology

Concerns:

- Object of study: are we describing human cognitive abilities, or a shared practice?
- Normative vs. descriptive: are we describing what people actually do, or what constitutes "correct" or "appropriate" behavior?
- Methods: to what extent are experimental methods – like cognitive task and protocol analyses – relevant to philosophy?

Logic and psychology

My own views:

- Philosophy of mathematics *should* interact with, and provide conceptual foundations for, fields that rely on some understanding of what it means to do mathematics:
  - mathematics itself
  - computer science
  - history of mathematics
  - psychology and cognitive science
  - education, pedagogy
- But, to be make progress on core issues, we have to be clear about the questions we are asking.
- In particular, it is often possible to disentangle epistemological issues from cognitive issues.
- A case needs to be made for overriding the default methodological separation.

Visualization in mathematics

Sample questions:

- Logical: what role does visualization and diagrammatic reasoning play in mathematics?
- Cognitive: how do we do it?
- Computational: how can we support it or emulate it?
- Historical: how did these uses arise and evolve?
- Pedagogical: how should we use visualization in teaching?
- ...

I will focus on the role of the diagram in Euclid's *Elements*.

The *Elements*

For more than two thousand years, Euclid's *Elements* was held to be the paradigm for rigorous argumentation.

But the nineteenth century raised concerns:

- Conclusions are drawn from diagrams, using "intuition" rather than precise rules.
- Particular diagrams are used to infer general results (without suitable justification).

Axiomatizations due to Pasch and Hilbert, and Tarksi's formal axiomatization later on, were thought to make Euclid rigorous.

The *Elements*

But in some ways, they are unsatisfactory.

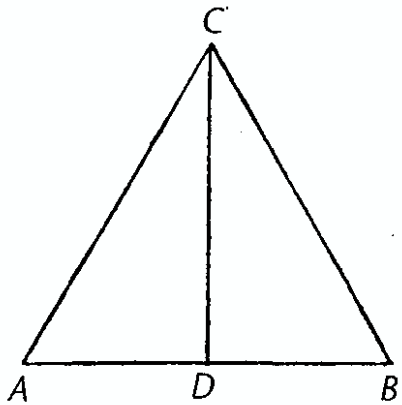
- Proofs in the new systems look very different from Euclid's.
- The initial criticisms belie the fact that Euclidean practice was remarkably stable for more than two thousand years.

Our project (Mumma, Dean, and me):

- Describe a formal system that is much more faithful to Euclid.
- Argue that the system is sound and complete (for the theorems it can express) relative to Euclidean fields.
- Show that the system can easily be implemented using contemporary automated reasoning technology.

Proposition 10

To bisect a given finite straight line.



Let AB be the given finite straight line.

Thus it is required to bisect the finite straight line AB.

Let the equilateral triangle ABC be constructed on it, [I. 1]
and let the angle ACB be bisected by the straight line CD; [I. 9]

I say that the straight line AB has been bisected at the point D.

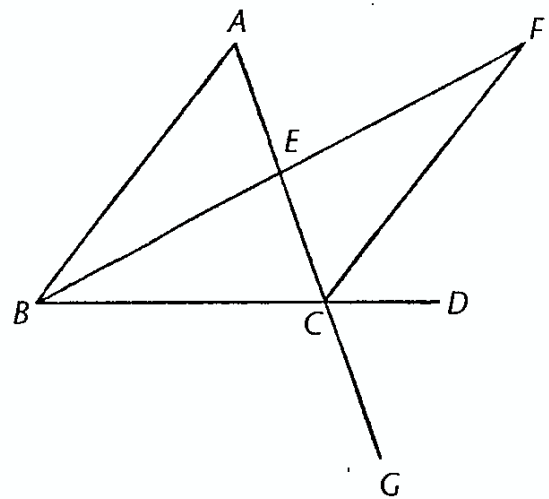
For, since AC is equal to CB, and CD is common,
the two sides AC, CD are equal to the two sides BC, CD respectively;
and the angle ACD is equal to the angle BCD;
therefore the base AD is equal to the base BD. [I. 4]

Therefore the given finite straight line AB has been bisected at D.

Q.E.F.

Proposition 16

In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.



Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*;
I say that the exterior angle *ACD* is greater than either of the interior and opposite angles *CBA, BAC*.

Let *AC* be bisected at *E*, **[I. 10]**
and let *BE* be joined and produced in a straight line to *F*;
let *EF* be made equal to *BE*, **[I. 3]**
let *FC* be joined, **[Post. 1]**
and let *AC* be drawn through to *G*. **[Post. 2]**

Then, since *AE* is equal to *EC*, and *BE* to *EF*,
the two sides *AE, EB* are equal to the two sides *CE, EF* respectively;
and the angle *AEB* is equal to the angle *FEC*, for they are vertical angles. **[I. 15]**

Therefore the base *AB* is equal to the base *FC*, and the triangle *ABE* is equal to the triangle *CFE*,
and the remaining angles are equal to the remaining angles respectively, namely those which the equal sides subtend; **[I. 4]**

therefore the angle *BAE* is equal to the angle *ECF*.

First salient feature: the use of diagrams

Observation: the diagram is inessential to the communication of the proof. (Rather, it is used to "see" that the inferences are correct.)

Exercise:

- Let p and q be points on a line.
- Let r be between p and q.
- Let s be between p and r.
- Let t be between r and q.

Is s necessarily between p and t?

Methodological stance: from a logical perspective, the way to characterize diagrammatic reasoning is in terms of the class of inferences that are licensed.

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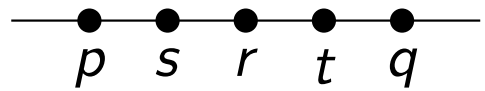

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First salient feature: the use of diagrams

Observation (Manders): In a Euclidean proof, diagrams are only used to infer "co-exact" (regional / topological) information, such as incidence, intersection, containment, etc.

Exact (metric) information, like congruence, is always made explicit in the text.

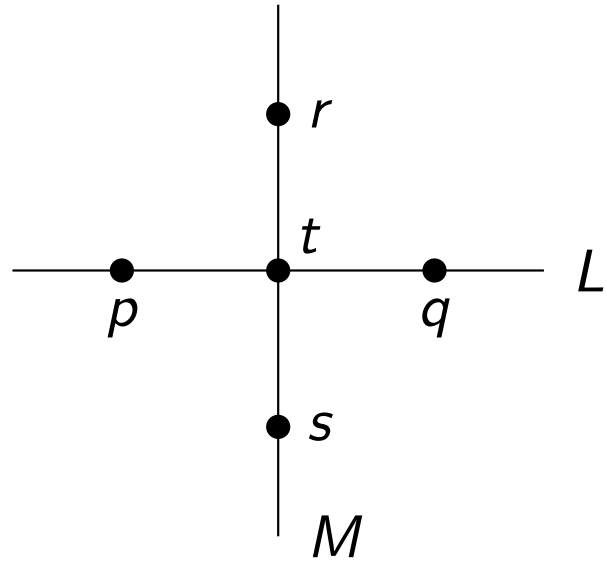
Poincaré: "Geometry is the art of precise reasoning from badly constructed diagrams."

Solution: take the "diagram" to be a representation of the relevant data.

Second salient feature: generality

Some aspects of diagrammatic inference are puzzling:

- Let p and q be distinct points.
- Let L be a line though p and q.
- Let r and s be points on opposite sides of L.
- Let M be the line through r and s.
- Let t be the intersection of L and M.



Is t necessarily between r and s? Is t necessarily between p and q?

The diagram was needed to "see" that L and M intersect. But not every feature found in a particular diagram is generally valid.

Euclid manages to avoid drawing invalid conclusions. We need an explanation as to what secures the generality.

Third salient feature: logical form

Theorems in Euclid are of the form:

> *Given points, lines, circles, satisfying . . . , there are points, lines, circles satisfying . . .*

where each . . . is a conjunction of literals.

(If the inner existential quantifier is absent, it is a "demonstration" rather than a "construction.")

Proofs contain a construction part, and a deduction part.

Reasoning is linear, assertions are literals.

Exceptions: proof by contradiction, using a case distinction (sometimes "without loss of generality").

Fourth salient feature: nondegeneracy

In the statement of a theorem, points are generally assumed to be distinct, triangles are nondegenerate, etc.

Two issues:

- Sometimes the theorem still holds in some degenerate cases.
- When the theorems are applied, Euclid doesn't always check nondegeneracy.

I will have little to say about this; in our system, nondegeneracy requirement are stated explicitly.

Formalizing Euclid

Prior efforts:

- Nathaniel Miller's Ph.D. thesis (2001): system is very complicated; generality is attained by considering cases exhaustively.
- John Mumma's Ph.D. thesis (2006): employs diagrams (and equivalence relation on diagrams); generality is attained using rules.

Our formal system, E, is derived from Mumma's. But now a "diagram" is nothing more than an abstract representation of topological information. The system spells out what can be inferred from the diagram.

The language of E

Basic sorts:

- diagram sorts: points $p, q, r, \ldots$, lines $L, M, N, \ldots$, circles $\alpha, \beta, \gamma, \ldots$
- metric sorts: lengths, angles, and areas.

Basic symbols:

- diagram relations: on(p, L), same-side(p, q, L), between(p, q, r), on(p, γ), inside(p, γ), center(p, γ), intersects(L, M), $=$
- metric functions and relations: $+$, $<$, $=$, right-angle
- connecting functions: $\overline{pq}$, $\angle pqr$, $\triangle pqr$

Other relations can be defined from these; e.g.

$$\text{diff-side}(p, q, L) \equiv \neg\text{on}(p, L) \wedge \neg\text{on}(q, L) \wedge \neg\text{same-side}(p, q, L)$$

Sequents

The proof system establishes sequents of the following form:

$$\Gamma \Rightarrow \exists \vec{q}, \vec{M}, \vec{\beta}.\ \Delta$$

where Γ and Δ are sets of literals.

Applying a construction rule or prior theorem augments $\vec{q}$, $\vec{M}$, $\vec{\beta}$, Δ.

Applying deductive inferences augments Δ.

Case splits and suppositional reasoning temporarily augment Γ.

I need to describe:

- Construction rules.
- Deductive inferences.

Diagram inferences are implicit in both.

Construction rules

"Let p be a point on L"
No prerequisites.

"Let p be a point distinct from q and r"
No prerequisites.

"Let L be the line through p and q"
Requires $p \neq q$.

"Let p be the intersection of L and M."
Requires that L and M intersect.

And so on...

Deductive inferences

Four types:

1. Diagram inferences: any fact that can be "read off" from the diagram.
2. Metric inferences: essentially linear arithmetic on lengths, angles, and areas.
3. Diagram to metric: for example, if q is between p and r, then $\overline{pq} + \overline{qr} = \overline{pr}$, and similarly for areas and angles.
4. Metric to diagram: for example, if p is the center of γ, q is on γ, and $\overline{pr} < \overline{pq}$, then r is inside γ.

Diagram inferences

Both construction inferences and diagram inferences require an account of what can be "read off" from the diagram.

We get this by closing the diagrammatic data in $\Gamma \cup \Delta$ under various rules, including:

- properties of "between"
- properties of "same side"
- "Pasch rules," relating "between" and "same side"
- triple incidence rules
- circle rules
- intersection rules

These yield conclusions that are generally valid, that is, common to all possible realizations.

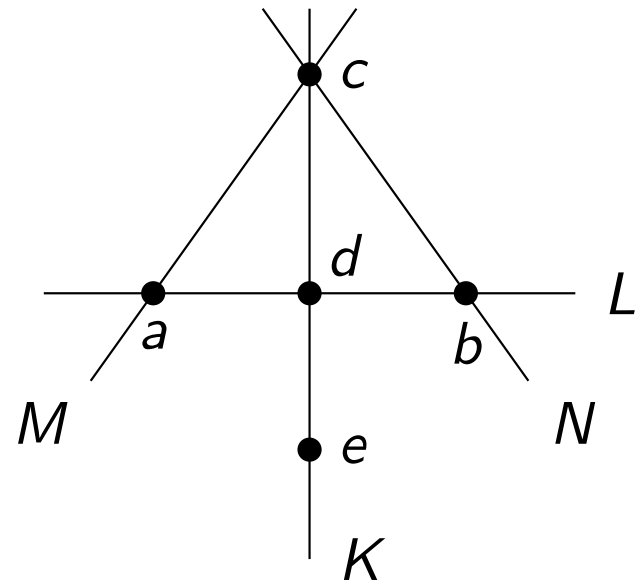
Proposition I.10. Assume a and b are distinct points on L. Construct a point d such that d is between a and b, and $\overline{ad} = \overline{db}$.

By Proposition I.1 applied to a and b, let c be a point such that $\overline{ab} = \overline{bc}$ and $\overline{bc} = \overline{ca}$ and c is not on L.

Let M be the line through c and a.

Let N be the line through c and b.

By Proposition I.9 applied to a, c, b, M, N, let e be a point such that $\angle ace = \angle bce$, b and e are on the same side of M, and a and e are on the same side of N.



Let K be the line through c and e.

Let d be the intersection of K and L.

Hence $\angle ace = \angle acd$.

Hence $\angle bce = \angle bcd$.

By Proposition I.4 applied to a, c, d, b, c, d have $\overline{ad} = \overline{bd}$.

Q.E.F.

Completeness

Tarski's first-order axiomatization of Euclidean geometry yields a complete theory of the Euclidean plane (inter-interpretable with real closed fields).

Drop the completeness axiom, and replace it with an axiom asserting that if a line L passes through a point inside a circle α, then L and α intersect.

The resulting theory is inter-interpretable with the theory of "Euclidean fields," and so is complete wrt "ruler and compass constructions." (Ziegler: it is also undecidable.)

Theorem. If a sequent of E is valid wrt to ruler and compass constructions, it can be derived in E.

Completeness

One strategy: interpret Tarski's theory in E.

Problem: Tarski includes full first-order logic!

Solution: With slight tinkering, Tarski's theory can be made "geometric," i.e. the axioms can be put in a restricted logical form.

A cut-elimination theorem due to Sara Negri then implies that any geometric assertion provable in Tarski's theory has a geometric proof.

Such a proof can be simulated in E.

Completeness

Outline of the proof:

1. Suppose a sequent A of E is valid for the intended semantics.
2. Then a translation $\pi(A)$ to Tarki's language is also valid for the intended semantics.
3. So it is provable in Tarski's theory.
4. So it has a cut-free proof.
5. This proof can be translated back to E, so E proves $\rho(\pi(A))$.
6. From this, E can derive the original sequent, A.

Implementation

Ben Northrop implemented the diagram inferences in Java with a saturation algorithm. But for moderately complex diagrams, the implementation is too slow to be of practical use.

We also tried first-order theorem provers, Spass and E, which do very well on the diagrammatic inferences (E does better).

Modern "satisfiability modulo theories" theorem provers allow one to prove universal assertions over mixed domains, including real linear arithmetic.

Alas, our diagram axioms are universal, which puts them outside the SMT framework.

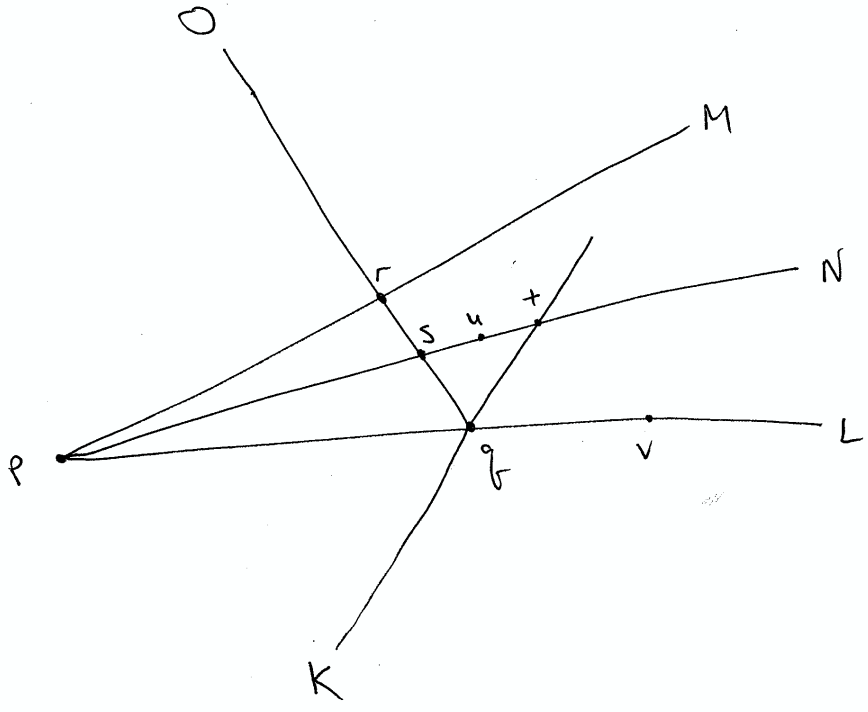
But some SMT solvers use heuristic instantiation of quantifiers.

Implementation

In a fairly complicated diagram, Z3 plows through all our inferences automatically. CVC3 also does pretty well.

In fact, they also handle all the metric inferences, as well as the ordinary propositional logic needed to handle case splits, proofs by contradiction, and so on.

In other words, SMT solvers can be used as a complete back end to check the inferences in E.



Data: all incidences (except on(u,N))

bet(p,s,T)

bet(q,s,r)

bet(s,u,T)

$p \neq q$

$\neg$ on(r,L)

```
:formula (sameside p t O)
:formula (sameside s t O)
:formula (not (sameside s t M))
:formula (not (sameside u t M))
:formula (bet s p t)
:formula (= M N)
:formula (bet q s u)
:formula (on q N)
:formula (= q t)
:formula (not (< (seg s u) (seg s t)))
:formula (not (< (seg u s) (seg s t)))
:formula (not (< (+ (seg s u) (seg u t)) (seg p t)))
:formula (not (< (+ (seg u s) (seg u t)) (seg p t)))
:formula (on u L)
:formula (on t L)
:formula (on p K)
:formula (not (sameside r s L))
:formula (not (sameside s u L))
:formula (not (sameside r u L))
:formula (sameside s v K)
:formula (not (= (+ (angle r p s) (angle s p q)) (angle r p q)))
:formula (not (sameside p s K))
:formula (not (sameside s t L))
:formula (= L K)
:formula (= q s)
:formula (= q t)
:formula (= q p)
:formula (not (= (+ (angle p q s) (angle s q t)) (angle p q t)))
:formula (not (< (angle p q s) (angle p q t)))
:formula (not (implies
  (= (+ (angle p q s) (angle s q t)) (angle p q t))
    (< (angle p q s) (angle p q t))))
```

Implementation

Using an SMT solver, it is easy to check Euclidean proofs:

When the user asserts a theorem: create the initial objects, assert hypotheses, and remember the conclusion.

When the user applies a construction rule: check prerequisites, create objects, assert properties.

When the user types "hence A": check A follows from the database, and if so, assert it explicitly.

For suppositional reasoning: push the state, assert the supposition, verify the conclusion, pop the state, and assert a conditional.

When the user types "QED," check that the negation of the theorem's conclusion is inconsistent.

Automated geometric reasoning

Approaches:

- Reduction to real closed fields, CAD: slow
- Wu's method: extremely powerful, but cannot handle order relations (like "between").
- Area method: extremely powerful, produces readable proofs, complete for the class of "constructive linear theorems," but once again cannot handle order.
- Synthetic methods.

Our approach falls into the last category, and, as far as automated reasoning goes, is fairly naive.

Automated geometric reasoning

See Chou, Gao, Zhang, "A deductive database approach to automated geometry theorem proving and discovering."

This provides an approach that is similar to ours, with much more sophisticated representations of diagrammatic information.

Our analysis does offer a broader lesson, though: an effective approach to formal verification is to combine more manageable domains (in our case, "diagram information" and "metric information") in principled ways.

Conclusions

Our modest claims:

- We have a clean analysis of the type of reasoning that is used in books I–IV of the *Elements*.
- Our system is sound and complete for the expected semantics.
- The analysis makes it easy to verify formal texts that are very close to proofs in the *Elements*.
- This provides a clear sense in which the *Elements* is more rigorous than commonly acknowledged.
- We have analyzed the *logical form* of diagrammatic inference, separating these questions from cognitive, pedagogical, and historical terms.
- The analysis can support further inquiry into *why* these inferences are basic to the practice.

References

See:

- Manders, Ken, *The Euclidean Diagram*
- Mumma's Ph.D. thesis, *Intuition Formalized: Ancient and Modern Methods of Proof in Elementary Geometry*
- Avigad, Dean, Mumma, *A formal system for Euclid's Elements*
- Avigad, review of Marcus Giaquinto, *Visual Thinking in Mathematics: An Epistemological Study*

Also, anything by John Mumma, such as:

- *Proofs, Pictures and Euclid*
- *Constructive Geometric Reasoning and Diagrams*