Inverting the Furstenberg correspondence

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Szemerédi's theorem

Theorem

For every k and $\delta > 0$, there is an n large enough, such that if A is any subset of $\{0, \ldots, n-1\}$ with density at least δ, then A has an arithmetic progression of length k.

In 1977, Furstenberg showed that this is a consequence of the following:

Theorem

For every k, measure-preserving system $(X, \mathcal{X}, \nu, U)$, and meausurable set E with $\nu(E) > 0$, there is an $i > 0$ such that $\nu(E \cap U^{-i}E \cap U^{-2i}E \cap \ldots \cap U^{-(k-1)i}E) > 0$.

The idea: by compactness, a sequence of counterexamples to Szemerédi's theorem yields a counterexample $(X, \mathcal{X}, \nu, U)$ and E.

Definitions

Let $n = \{0, \ldots, n-1\}$. Identify subsets of n with binary sequences. For example, 0110101100 denotes $\{1, 2, 4, 6, 7\} \subseteq 10$.

For $A \subseteq n$, let $D_A(\sigma)$ be the number of times σ occurs in A (allowing wraparound), divided by n.

Let 2^ω denote Cantor space, with Borel sets $\mathcal{B}$.

Let $[\sigma] = \{f \mid \sigma \subset f\}$. So $\mathcal{B}$ is generated by $\{[\sigma] \mid \sigma \in 2^{<\omega}\}$.

Let T denote the shift-left map on 2^ω, $(Tf)(n) = f(n+1)$.

For every A, $\mu_A([\sigma]) = D_A(\sigma)$ defines a T-invariant measure.

But the Furstenberg recurrence theorem is trivially true for $(2^\omega, \mathcal{B}, \mu_A, T)$.

The Furstenberg correspondence principle

Theorem

Given (A_n) with $A_n \subseteq n$, there are a T-invariant measure μ on 2^ω and a subsequence (A_{n_i}) such that

$$\mu([\sigma]) = \lim_{i \to \infty} D_{A_{n_i}}(\sigma),$$

for every σ.

Proof: let μ be a limit point of the set $\{\mu_{A_i}\}$ in the vague topology.

In more elementary terms: iteratively thin out the original sequence to ensure that each $D_{A_{n_i}}(\sigma)$ converges.

(So apply the measure-theoretic statement with $E = [1]$.)

Inverting the correspondence

In fact, any shift-invariant μ on 2^ω arises in this way.

Theorem

Let μ be any T-invariant measure on 2^ω. Then for each j and ε, there are an $n \leq 2^{O(j/\varepsilon)}$ and an $A \subseteq n$, such that for every σ of length at most j,

$$|\mu([\sigma]) - D_A(\sigma)| < \varepsilon.$$

Moreover, if n is sufficiently large (independent of μ), there is always a set A with this property.

Inverting the correspondence

To prove this, fix j and $\varepsilon > 0$, choose k large, and consider $\{[\tau] \mid len(\tau) = k\}$.

$$\begin{aligned}
\mu([\sigma]) &= \frac{1}{k} \sum_{i<k} \mu(T^{-i}[\sigma]) \\
&= \frac{1}{k} \sum_{i<k} \sum_{\tau} \mu(T^{-i}[\sigma] \cap \tau) \\
&= \sum_{\tau} \frac{1}{k} \sum_{i<k} \mu(T^{-i}[\sigma] \cap \tau) \\
&= \sum_{\tau} \frac{1}{k} (N_\sigma(\tau) + O(j))\mu(\tau) \\
&= \sum_{\tau} D_\sigma(\tau)\mu(\tau) + O(j/k).
\end{aligned}$$

Build A by concatenating τ's in the right proportion.

A uniform version of the recurrence theorem

Theorem
For every k, $(X, \mathcal{X}, \nu, U)$, and E with $\nu(E) > 0$, there is an $i > 0$ such that $\nu(E \cap U^{-i}E \cap U^{-2i}E \cap \ldots \cap U^{-(k-1)i}E) > 0$.

Theorem
For every k and $\delta > 0$, there are n and $\eta > 0$ such that for every $(X, \mathcal{X}, \nu, U)$ and E with $\nu(E) \geq \delta$, there is an i such that $0 < i \leq n$ and $\nu(E \cap U^{-i}E \cap U^{-2i}E \cap \ldots \cap U^{(k-1)i}E) \geq \eta$.

In the second, η and n depend only on k and $\delta > 0$, and not $(X, \mathcal{X}, \nu, U)$ or E.

Versions of Szemerédi's theorem

Thus we have three versions of the theorem:

1. the finitary one
2. the measure-theoretic one
3. the uniform measure-theoretic one

1 and 3 can be proved equivalent *without compactness*.

1 $\Rightarrow$ 3: use a finite approximation to the measure.

3 $\Rightarrow$ 1: specialize the measure to μ_A, for a finite set A.

On the other hand, one can use compactness to pass from 2 to *either* 1 or 3.

The moral

$$\text{study of patterns in } \mathbb{Z}/n\mathbb{Z}$$
$$=$$
$$\text{uniform, complexity bounded ergodic theory}$$

Compactness is not needed to mediate the passage from finite to infinite, but, rather, to obtain uniform bounds on complexity.

The bad news: one cannot always bound the complexity of an ergodic-theoretic construction.

Unpredictably noisy limits

Let $\mathcal{B}_k$ be the factor of 2^ω generated by $\{[\sigma] \mid len(\sigma) = k\}$.

Say $\mathcal{B}_k$-measurable sets and functions are *simple*, with *complexity at most* k.

Proposition

Let μ be the uniform measure on 2^ω. There is an f such that

- *f is the limit of a computable sequence of simple functions f_n.*
- *There is no computable bound on the rate of convergence of $E(f|\mathcal{B}_k)$ to f.*

Unpredictably noisy ergodic limits

Theorem (V'yugin)

There is a computable shift-invariant measure μ on 2^ω such there is no computable bound on the rate of convergence of $A_n 1_{[1]}$.

V'yugin's construction *doesn't* have the property on the previous slide. But:

Theorem

There is a computable shift-invariant measure μ on 2^ω such that if $f = \lim_n A_n 1_{[1]}$, there is no computable bound on the rate of convergence of $E(f|\mathcal{B}_k)$ to f.

Corollary

There is no bound on the complexity of f that is uniform in μ.

Final observations

Tao has presented an quantitative ergodic-theoretic proof of Szemerédi's theorem in the language of $\mathbb{Z}/n\mathbb{Z}$.

The proof can be viewed as a uniform, complexity bounded version of a (by now) straightforward ergodic-theoretic argument, specialized to finite measures μ_A.

The arguments used to obtain the necessary uniformities are, however, quite delicate and subtle.

Given that these uniformities form the nexus between infinitary and discrete methods, it seems important to understand how and when they can be obtained in the ergodic-theoretic setting.