Machine learning, formal methods, and mathematics

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AI to Assist Mathematical Reasoning Workshop

June 13, 2023

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Symbolic methods and neural methods

Symbolic methods (good old fashioned AI):

- logic-based representations
- precise, exact
- explicit rules of inference

Neural methods (machine learning)

- distributed representations
- probabilistic, approximate
- based on lots of data

Mathematics needs both:

- neural methods can discover patterns and connections
- symbolic methods can help us get the answers right

Symbolic methods and neural methods

Mathematical knowledge:

- The sum of the positive integers up to 100 is 5,050.
- For $n > 2$, there are no integer solutions to $x^n + y^n = z^n$ with all of x, y, and z nonzero.

Empirical knowledge:

- It is likely to rain tomorrow.
- Raising interest rates is likely to lead to a recession.
- Jones is not likely to default a loan.

These are all things we might want to know.

Symbolic methods and neural methods

Mathematics often gives us ways of being precise about imprecise knowledge.

For example, we may extract a model from empirical data and reason about the model.

- The model may be only probably approximately correct.
- But we can reason precisely about the evidence for it and the implications.

We need this type of reasoning to think through the consequences of our actions, deliberate, and plan.

Formal methods in mathematics

Formal methods are a body of logic-based methods used in computer science to

- write specifications for hardware, software, protocols, and so on, and
- verify that artifacts meet their specifications.

The same technology is useful for mathematics.

I use "formal methods in mathematics" and "symbolic AI for mathematics" roughly interchangeably.

Formal methods in mathematics

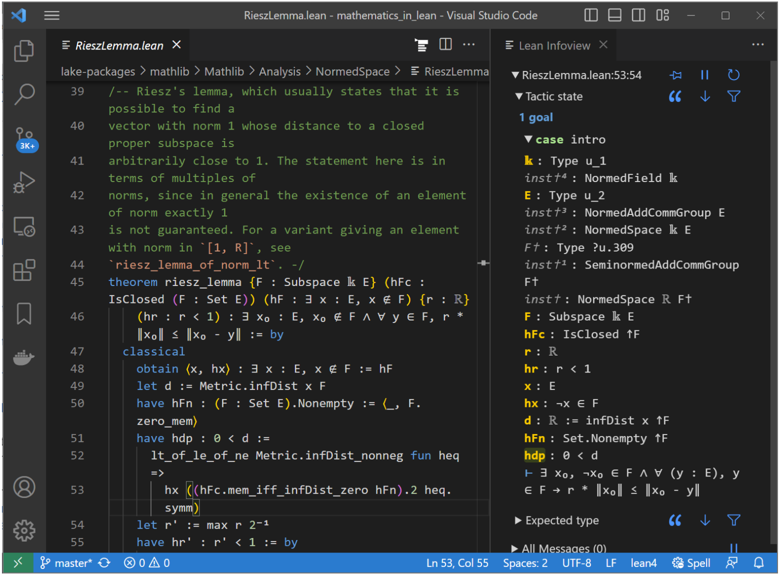
Since the early twentieth century, we have known that mathematics can be represented in formal axiomatic systems.

Computational "proof assistants" allow us to write mathematical definitions, theorems, and proofs in such a way that they can be

- processed,
- verified,
- shared, and
- searched

by mechanical means.

Formal methods in mathematics



Formal methods in mathematics

Some talks (with links):

- Thomas Hales, Big Conjectures
- Sébastien Gouëzel, On a Mathematician's Attempts to Formalize his Own Research in Proof Assistants
- Patrick Massot, Why Explain Mathematics to Computers?
- Kevin Buzzard, The Rise of Formalism in Mathematics
- Johan Commelin, Abstract Formalities
- Adam Topaz, The Liquid Tensor Experiment
- Heather Macbeth, Algorithm and Abstraction in Formal Mathematics

Formal methods in mathematics

MATHEMATICS AND THE FORMAL TURN

JEREMY AVIGAD

ABSTRACT. Since the early twentieth century, it has been understood that mathematical definitions and proofs can be represented in formal systems systems with precise grammars and rules of use. Building on such foundations, computational proof assistants now make it possible to encode mathematical knowledge in digital form. This article enumerates some of the ways that these and related technologies can help us do mathematics.

INTRODUCTION

One of the most striking contributions of modern logic is its demonstration that mathematical definitions and proofs can be represented in formal axiomatic systems. Among the earliest were Zermelo's axiomatization of set theory, which was introduced in 1908, and the system of ramified type theory, which was presented by Russell and Whitehead in the first volume of *Principia Mathematica* in 1911. These were so successful that Kurt Gödel began his famous 1931 paper on the incompleteness theorems with the observation that "in them all methods of proof used today in mathematics are formalized, that is, reduced to a few axioms and rules of inference." Cast in this light, Gödel's results are unnerving: no matter what mathematical methods we subscribe to now or at any point in the future, there will always be mathematical questions, even ones about the integers, that cannot be settled on that basis—unless the methods are in fact inconsistent. But the positive

Formal methods in mathematics

Executive summary: formal methods can be useful for

- verifying theorems
- correcting mistakes
- gaining insight
- building libraries
- searching for definitions and theorems
- refactoring proofs
- refactoring libraries
- engineering concepts
- communicating
- collaborating
- managing complexity
- managing the literature
- teaching
- improving access
- using mathematical computation
- using automated reasoning
- using AI

The technology holds a lot of promise.

Formal methods and AI

Applications of machine learning to mathematics are a new frontier.

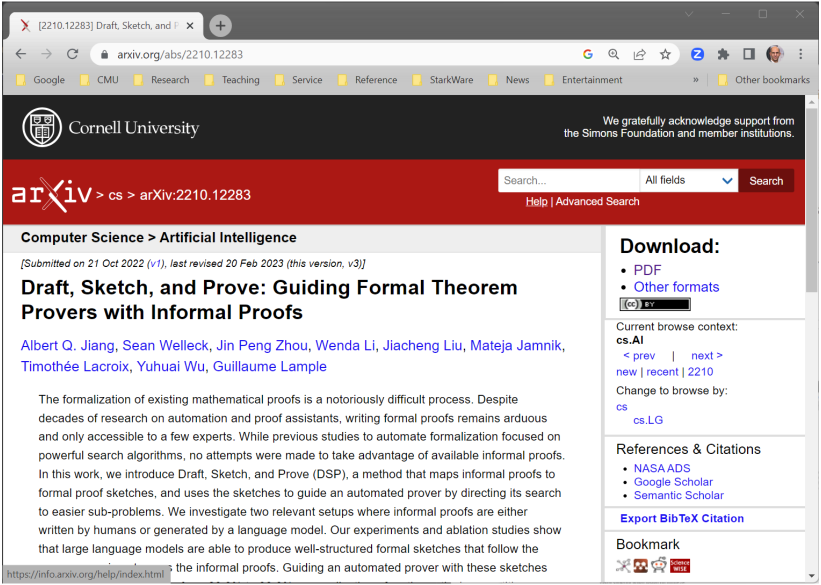
There have been important machine-learning projects using Mizar, HOL Light, Metamath, Isabelle, Coq, Lean, and others.

“Draft, sketch, and prove” combines neural and symbolic methods:

- First, a large language model drafts an informal proof.
- Then it sketches a formal proof.
- Automated reasoners fill in the details and verify that they are correct.

Searching for formally checkable content provides a clear signal.

Formal methods and AI



[2210.12283] Draft, Sketch, and P

arxiv.org/abs/2210.12283

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We gratefully acknowledge support from the Simons Foundation and member institutions.

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[Submitted on 21 Oct 2022 (v1), last revised 20 Feb 2023 (this version, v3)]

Draft, Sketch, and Prove: Guiding Formal Theorem Provers with Informal Proofs

Albert Q. Jiang, Sean Welleck, Jin Peng Zhou, Wenda Li, Jiacheng Liu, Mateja Jamnik, Timothée Lacroix, Yuhuai Wu, Guillaume Lample

The formalization of existing mathematical proofs is a notoriously difficult process. Despite decades of research on automation and proof assistants, writing formal proofs remains arduous and only accessible to a few experts. While previous studies to automate formalization focused on powerful search algorithms, no attempts were made to take advantage of available informal proofs. In this work, we introduce Draft, Sketch, and Prove (DSP), a method that maps informal proofs to formal proof sketches, and uses the sketches to guide an automated prover by directing its search to easier sub-problems. We investigate two relevant setups where informal proofs are either written by humans or generated by a language model. Our experiments and ablation studies show that large language models are able to produce well-structured formal sketches that follow the ... the informal proofs. Guiding an automated prover with these sketches

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Mathematics and computer science

Mathematicians and computer scientists have different skill sets and outlooks.

Mathematicians enjoy:

- solving hard problems
- finding patterns
- finding deep connections
- developing powerful abstractions.

Computer scientists enjoy:

- implementing complex systems
- finding clever optimizations
- making systems more reliable and robust.

Mathematics and computer science need each other.

Cooperation and collaboration

Digital technology provides new platforms for cooperation and collaboration.

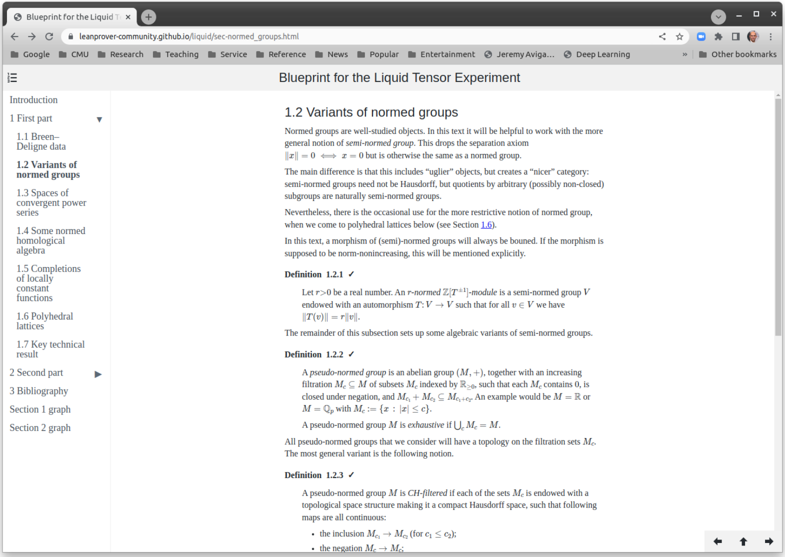
Communities of practitioners use social media to:

- ask questions and get help
- pose challenges to one another
- make plans and coordinate efforts.

The Liquid Tensor Experiment is a good model:

- The formalization was in kept in a shared online repository.
- Participants followed an informal blueprint with links to the repository.
- Participants were in constant contact on Zulip.
- A proof assistant made sure the pieces fit together.

Cooperation and collaboration



leanprover-community.github.io/liquid/sec-normed_groups.html

Blueprint for the Liquid Tensor Experiment

1.2 Variants of normed groups

Normed groups are well-studied objects. In this text it will be helpful to work with the more general notion of *semi-normed group*. This drops the separation axiom $\|x\| = 0 \iff x = 0$ but is otherwise the same as a normed group.

The main difference is that this includes “uglier” objects, but creates a “nicer” category: semi-normed groups need not be Hausdorff, but quotients by arbitrary (possibly non-closed) subgroups are naturally semi-normed groups.

Nevertheless, there is the occasional use for the more restrictive notion of normed group, when we come to polyhedral lattices below (see Section 1.6).

In this text, a morphism of (semi)-normed groups will always be bouned. If the morphism is supposed to be norm-nonincreasing, this will be mentioned explicitly.

Definition 1.2.1 ✓

Let $r>0$ be a real number. An *r-normed $\mathbb{Z}[T^{\pm 1}]$-module* is a semi-normed group V endowed with an automorphism $T\colon V \to V$ such that for all $v \in V$ we have $\|T(v)\| = r\|v\|$.

The remainder of this subsection sets up some algebraic variants of semi-normed groups.

Definition 1.2.2 ✓

A *pseudo-normed group* is an abelian group $(M, +)$, together with an increasing filtration $M_c \subseteq M$ of subsets M_c indexed by $\mathbb{R}_{\geq 0}$, such that each M_c contains 0, is closed under negation, and $M_{c_1} + M_{c_2} \subseteq M_{c_1+c_2}$. An example would be $M = \mathbb{R}$ or $M = \mathbb{Q}_p$ with $M_c := \{x : |x| \leq c\}$.

A pseudo-normed group M is *exhaustive* if $\bigcup_c M_c = M$.

All pseudo-normed groups that we consider will have a topology on the filtration sets M_c. The most general variant is the following notion.

Definition 1.2.3 ✓

A pseudo-normed group M is *CH-filtered* if each of the sets M_c is endowed with a topological space structure making it a compact Hausdorff space, such that following maps are all continuous:

- the inclusion $M_{c_1} \to M_{c_2}$ (for $c_1 \leq c_2$);
- the negation $M_c \to M_c$;

Cooperation and collaboration

The port of Mathlib, a large formal mathematical library, is another example.

Since 2016, the community has been using Lean 3. We are now just beginning to use Lean 4, which is not backward compatible.

The Lean 3 library has over a million lines of formal proof.

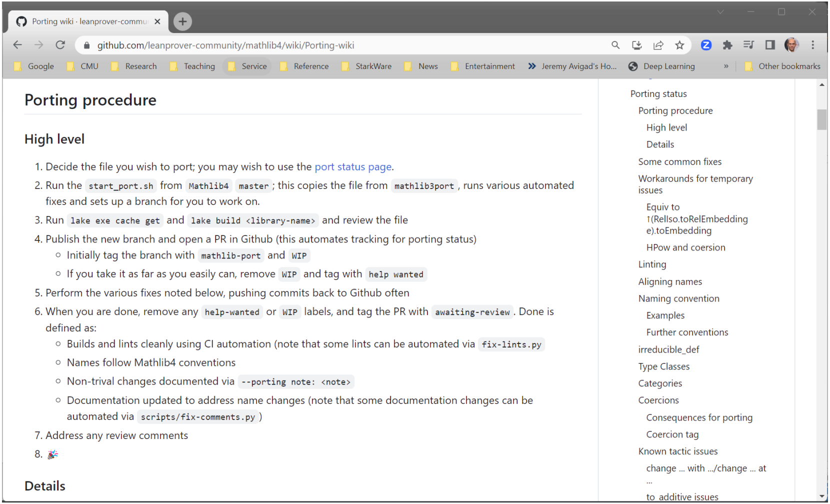
Cooperation and collaboration

How do you port a million lines of formal proof?

- Mario Carneiro wrote an automatic translator that comes close, but needs user intervention.
- About 40,000 lines of tactics — small scale automation — had to be rewritten entirely.
- Carneiro and Scott Morrison are leading a team of volunteers.
- There were months of planning and discussion.
- The effort requires repairing translations manually and adapting to changes in automation.
- Contributions to the old library have continued.
- There has been an endless stream of problems to address.

Cooperation and collaboration

Cooperation and collaboration



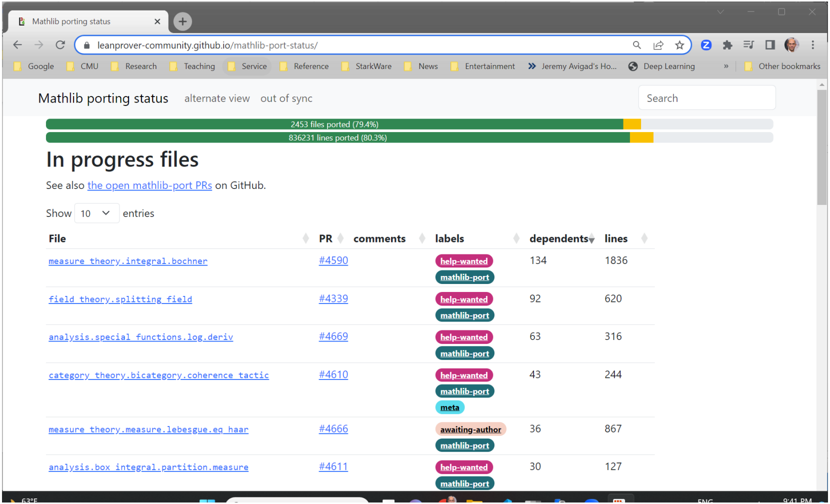
Porting procedure

High level

1. Decide the file you wish to port; you may wish to use the port status page.
2. Run the start_port.sh from Mathlib4 master; this copies the file from mathlib3port, runs various automated fixes and sets up a branch for you to work on.
3. Run lake exe cache get and lake build <library-name> and review the file
4. Publish the new branch and open a PR in Github (this automates tracking for porting status)
   - Initially tag the branch with mathlib-port and WIP
   - If you take it as far as you easily can, remove WIP and tag with help wanted
5. Perform the various fixes noted below, pushing commits back to Github often
6. When you are done, remove any help-wanted or WIP labels, and tag the PR with awaiting-review. Done is defined as:
   - Builds and lints cleanly using CI automation (note that some lints can be automated via fix-lints.py
   - Names follow Mathlib4 conventions
   - Non-trival changes documented via --porting note: <note>
   - Documentation updated to address name changes (note that some documentation changes can be automated via scripts/fix-comments.py)
7. Address any review comments
8. 🎉

Details

Cooperation and collaboration



Mathlib porting status   alternate view   out of sync   Search

2453 files ported (79.4%)

836231 lines ported (80.3%)

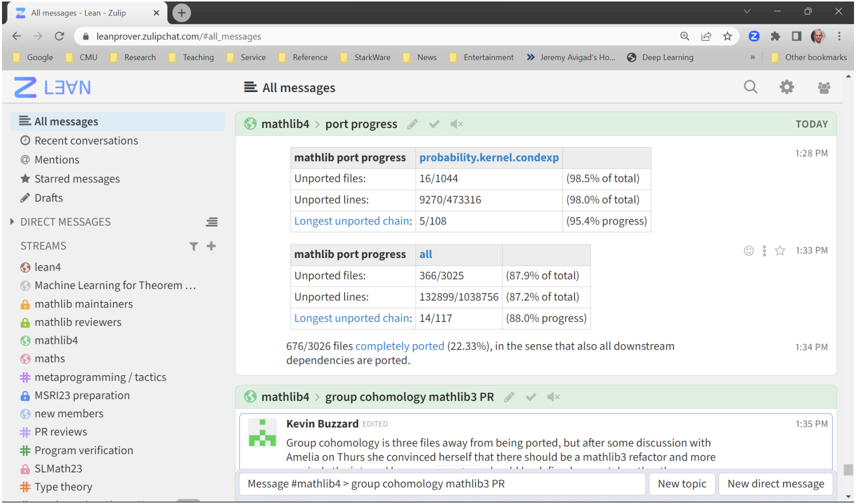
In progress files

See also the open mathlib-port PRs on GitHub.

Show 10 entries

File	PR	comments	labels	dependents	lines
measure_theory.integral.bochner	#4590		help-wanted mathlib-port	134	1836
field_theory.splitting_field	#4339		help-wanted mathlib-port	92	620
analysis.special_functions.log.deriv	#4669		help-wanted mathlib-port	63	316
category_theory.bicategory.coherence_tactic	#4610		help-wanted mathlib-port meta	43	244
measure_theory.measure.lebesgue.eq_haar	#4666		awaiting-author mathlib-port	36	867
analysis.box_integral.partition.measure	#4611		help-wanted mathlib-port	30	127

Cooperation and collaboration



Institutional challenges

Main challenges:

- Industrial research has to answer to corporate interests.
- Academic environments encourage specialization.

Neither is aligned with developing technology for mathematical research.

Academia is governed by traditional means of assessment:

- Mathematicians are evaluated by the judgments of experts and publication in top journals.
- Computer scientists are judged by citation counts, which are a proxy for impact.

Institutional challenges

There's a chicken and egg problem:

Computer scientists *can* get credit for developing useful technology for mathematics by showing that mathematicians are using it.

Mathematician's can't get credit for using technology unless they use it to make progress on traditional problems.

Both communities need to make substantial investments of time and energy before either will have anything to show for it.

Institutional challenges

Some of the contributors to the mathlib port:

Yury Kudryashov
Gabriel Ebner
Jason Yuen
Jeremy Tan Jie Rui
Moritz Doll
David Renshaw
Johan Commelin
Jon Eugster
Riccardo Brasca
Adam Topaz
Yakov Pechersky
Jakob von Raumer
Henrik Böving
Maxwell Thum
Richard Osborn
Scott Morrison
Mario Carneiro
Joël Riou
Eric Wieser
Jireh Loreaux
Arien Malec
Lukas Miaskiwskyi
Kevin Buzzard
Kyle Miller
Jujian Zhang
Alex Best
Reid Barton
Siddhartha Gadgil
Zachary Battleman
Ruben Van de Velde
Chris Hughes
Moritz Firsching
Matthew Ballard
Floris van Doorn
Yaël Dillies
Xavier Roblot
Heather Macbeth
Arthur Paulino
Frédéric Dupuis
Violeta Hernández
Anatole Dedecker
Winston Yin
Eric Rodriguez

Institutional challenges

We need to understand how to incentivize and reward:

- ongoing system development and maintenance
- ongoing library development and maintenance
- development and maintenance of web pages and collaboration platforms
- answering questions and training new users.

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- formal methods in mathematics
- formal methods and AI
- mathematics and computer science
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- institutional challenges

Summary

Progress on AI for mathematics requires input from three distinct communities:

- computer scientists working in formal methods (proof assistants, automated reasoning)
- computer scientists working in machine learning (large language models, reinforcement learning, and so on)
- mathematicians figuring out how to use the technology to do mathematics

These communities are for the most part disjoint and have disjoint expertise.

Summary

Progress in AI-assisted mathematics is going to require working together:

- Symbolic methods are good at computation, verification, and search, but struggles with combinatorial explosion and heterogeneous reasoning.
- Neural methods can gather, process, and synthesize huge amounts of data, but struggle to get the details right.
- Mathematicians understand the mathematics, and computer scientists won't get anywhere without that.

Summary

- Progress in AI for mathematics needs a combination of neural and symbolic methods.
- It also requires mathematicians and computer scientists working together.
- Advances in technology for mathematics require new forms of collaboration and interaction, and, at the same time, provide new means and platforms for that.
- We need better institutional support for collaborative, cross-disciplinary work and we need ways of assessing new kinds of mathematical contributions.