William Tait. **The provenance of pure reason: essays in the philosophy of mathematics and its history.** Oxford University Press, Oxford, 2005 x + 332 pp.

Plato has a bad reputation these days: the word "Platonism" is usually accompanied by the word "naive," and the notion that mathematical discourse has meaning in virtue of reference to nonspacial, atemporal objects is judged to be convenient but ultimately untenable. Moreover, the views of mathematics that currently dominate the analytic tradition are variants of Quinean empiricism, a position that is more closely linked to that of Plato's rival, Aristotle. On such views, mathematical claims depend on experience for justification in an indirect but continuous manner.

The problem with this picture is that it fails to account for the distinct character of mathematics vis-à-vis the empirical sciences. Quine's arguments that there is no sharp distinction between analytic and synthetic knowledge does not absolve us of the philosophical task of making sense of methodological differences that are glaringly salient in practice. It is therefore encouraging to read a collection of essays that is based on a conception of mathematics that is more faithful to the subject.

At the risk of oversimplifying, Tait's central views can be summarized as follows. First, the only viable foundation for mathematics is that afforded by the axiomatic conception. Whatever basis mathematics has in intuition and experience, it only becomes mathematics *per se* when we pass from these to first principles, that is, the primitive concepts, definitions, and axioms that form the basis for mathematical reasoning.

Second, this axiomatic conception makes it possible to distinguish between questions that are internal to mathematics, and those that are external. From an internal point of view, the question "do numbers exist?" is trivially answered in the affirmative. From an external point of view, the question is close to nonsensical; at least, it presupposes that there are extra-mathematical (Tait calls them "superrealist") grounds that could serve to provide an answer, and such grounds are far from apparent. Similarly, uses of the notion of "truth" within mathematics are invariably benign and easily understood in a deflationary sense (excluding, of course, the model-theoretic notion of truth, which is simply a formal mathematical notion). In contrast, Tait seems to feel that philosophers' external notions of truth have had little to add to our understanding of mathematics other than a good deal of unnecessary confusion.

Finally, the axiomatic conception also makes it possible to distinguish between internal and external evaluations of a theory:

> . . . even in its application to empirical science, the value of mathematics lies precisely in its separation from empirical matters, in its being entirely a creature of reason. For only with this separation can we distinguish between the question of the adequacy and coherence of the mathematical theory, in itself, in terms of which we wish to understand the phenomena and the question, given its coherence, of whether or not the rational structure it describes is the right one for

> the phenomena at issue. (page 5)

The phrase "given its coherence" is indicative of Tait's skepticism that there is a substantial philosophical project in the latter. He takes this to support a hands-off attitude towards development of mathematics, as advocated by Cantor in his oft-quoted passage on mathematics and freedom.

The views just described are laid out most clearly in Chapter 3, *Truth and Proof: The Platonism of Mathematics*, which is one of Tait's most important works; and in a sequel, *Beyond the Axioms: The Question of Objectivity in Mathematics*, which appears as Chapter 4. But they are the driving force behind all the essays, and serve to unify the diverse set of topics they treat. The first six essays explicate various aspects of mathematics including finitism, the axiom of choice, the law of the excluded middle, and set-theoretic reflection principles. The last six essays interpret the views of a number of important historical figures in the philosophy of mathematics, including Plato, Cantor, Frege, Hilbert, Wittgenstein, and Gödel. For the most part, these interpretations are seen to support the conception of mathematics described above. As one would expect from a logician of Tait's stature, all the essays are informed by a sound formal understanding.

The centrality of proof and axiomatic reasoning endears Tait's account to proof theorists; his position is, in large part, an articulation of the understanding of mathematics that forms the basis for proof-theoretic research in the foundations of mathematics. It is therefore inevitable that some will label the conception a brand of formalism, in a pejorative sense. Tait is sensitive to this, and has taken pains to argue, throughout his work, that his characterization is not only philosophically coherent and faithful to mathematical practice, but, further, does not deny the essential open-endedness of mathematics or its role with respect to the empirical sciences. Indeed, he sees Wittgenstein as having solved the problem of reconciling the axiomatic conception with a reasoned empiricism:

> Only empirical explanation is possible for why we have come to accept the basic principles that we do and why we apply them as we do—for why we have mathematics and why it is at it is. But it is only within the framework of mathematics as determined by this practice that we can speak of mathematical necessity. In this sense, which I believe Wittgenstein was first to fully grasp, mathematical necessity rides on the back of empirical contingency. (page 116)

Tait essentially takes Wittgenstein's process of "hardening to a rule" to describe the dialectic process that leads to axiomatic first principles; and he takes Wittgenstein's characterization of mathematics as a "form of life" to be life within an axiomatic framework. The fact that the excerpt just quoted appears almost as an aside in an essay on the axiom of choice is characteristic of the tenor of these essays. So, too, is the conclusion that he draws from the observations:

> Notice that this view of things leaves no room for so-called epistemological foundations of mathematics or for "foundations of mathemat-

> ics" in the sense of attempting to show that mathematics is "true" as opposed to showing that, in mathematics, a particular proposition is true. (*ibid.*)

Tait's essays are a pleasure to read. They are crisp, intelligent, compelling, and engagingly cantankerous. Space does not allow me to discuss them in detail, but they all merit serious consideration, including his analysis of the relationship between the axiom of choice and the law of the excluded middle; his analysis of reflection principles in set theory; his favorable interpretation of Plato's theory of forms (which encourages us to dismiss common contemporary readings that "attribute to him views that would have been as foolish or unintelligible in his time as they are in ours"); his readings of Cantor, Frege, and Wittgenstein; and his discussion of the unpublished writings collected in the third volume of Gödel's *Collected Works*. Instead, I will say only a few words about two of Tait's most longstanding interests, finitism and type theory.

Hilbert originally hoped to prove the consistency of infinitary mathematics using only finitary methods. Tait took up the latter notion in a 1981 essay, *Finitism*, which is reproduced in Chapter 1, and a 2002 follow-up, *Remarks on Finitism*, which is reproduced in Chapter 2. His goal was to provide an explication of finitism that is, first, independently interesting and defensible, and, second, consistent with the remarks in Hilbert and Bernays's two-volume *Grundlagen der Mathematik*. Tait has famously defended the formal system known as primitive recursive arithmetic, *PRA*, as the only analysis that meets both standards. Of the two goals, the second is the more nebulous; Hilbert and Bernays never articulated a clear position, and they had to adapt their views to Gödel's discovery of the incompleteness theorems between the writing of the two volumes. To date, however, there has been no conceptual analysis as compelling as Tait's, and although an appendix to Chapters 1 and 2 revises some of his claims as to Hilbert and Bernays's historical views, his commitment to *PRA* as the best conceptual explication of finitism has remained firm.

I will refer the reader to Tait's essays for the full analysis, and simply comment on an interesting issue that arises in passing. Hilbert's goal was to reduce modern mathematics to an epistemologically privileged fragment; in particular, he took the finitary objects to be "surveyable" in all their parts, thereby making it possible to ground finitary mathematics in a faculty of intuition. Tait, interestingly, challenges this assessment. He does, of course, allow that the concept of number is fundamental to most branches of mathematics, and this certainly accords elementary number theory a special status. But his argument casts doubt on whether there is an independent epistemic standard under which the concept of number is so privileged, and rightly emphasizes that stock appeals to a faculty of intuition fall short.

A number of essays in the book employ the "type-theoretic" characterization of mathematics, of which Tait himself was an early proponent, along with William Howard and Per Martin-Löf. On this view, mathematical objects always bear appropriate "types," and these, in turn, are characterized by rules that tell one how objects of that type are to be constructed and used in mathematical discourse. A further component of this view is that any mathematical proposition itself can be viewed as a type, namely, the type of objects that constitute a proof of the proposition. Both formal and philosophical considerations encourage one to view such types as datatype specifications, and the corresponding objects as computational data. As a result, the type-theoretic "propositions-as-types" framework is held by many to explicate constructive mathematics.

Tait is unusual in using type theory to explicate a *classical* mathematical standpoint, and he suggests the term "construction-theoretic" to characterize such a broader notion of constructivity. In some ways, the resulting characterization of classical mathematics is awkward. For example, in Chapter 5, Tait invests a good deal of effort in explaining why, on a construction-theoretic point of view, it is reasonable to postulate objects of type $\neg\neg A \to A$ for every proposition A. Another difficulty (and one which, in my view, Tait does not sufficiently address) has to do with extensionality and dependent types. Roughly, the question as to whether a (presentation of) an object has a certain type becomes undecidable unless one is careful as to the types of reasoning about equality that can play a role in type judgments; the necessary restrictions, while reasonable from a computational/constructive standpoint, seem artificial from a classical perspective. In short, the set-theoretic formulation of classical mathematics is in some ways more natural, and it is not clear what advantages the type-theoretic formulation has to offer.

Both the conceptual and historical analyses in this collection contain a wealth of insight. Tait's dismissals of alternative views is, however, sometimes too glib. For example, while skeptical that an external notion of truth has any role to play in the philosophy of mathematics, in selecting Reuben Hersch as a prototypical realist he has chosen an easy target; contemporary versions of naturalism at least purport to provide more robust notion of truths against which mathematical axioms can be measured. He similarly has little to say about the internal theoretical virtues that a mathematician might appeal to. One gets the sense that, on Tait's view, the proper role of the contemporary philosopher of mathematics is to clear away the metaphysical cobwebs and junk we have accumulated since Plato's time, and then step aside to let mathematics flourish. But viewing mathematics as a self-sufficient "form of life" does not inherently give us the conceptual wherewithal to engage in the practice reflectively, to evaluate our choice of axioms, or to cope with methodological and foundational disagreements when they arise. Moreover, as of late, some have expressed hope that it is possible to make sense of evaluatory claims in mathematics that go beyond judgments of correctness; for example, one might hope to understand what it is that makes a mathematical concept fruitful, powerful, or natural. An overly narrow focus on the axiomatic characterization of mathematics is not conducive to such a program.

Ultimately, however, where Tait is skeptical as to the role that philosophy can play, his skepticism is justified. His challenges to the philosophical community are therefore sharp and to the point. What this collection of essays provides is a lean, no-nonsense view of mathematics that can form the basis for substantive inquiry. The resulting conception of the subject is one that no serious philosopher of mathematics can ignore.

JEREMY AVIGAD, Department of Philosophy, Baker Hall 135, Carnegie Mellon University, Pittsburgh, PA 15213. avigad@cmu.edu.