Thomas Hales. **Dense Sphere Packings: A Blueprint for Formal Proofs.** Cambridge University Press, Cambridge, 2012 xiv + 271 pp.

In 1611, Johannes Kepler asserted that the highest density that can be achieved when arranging infinitely many congruent spheres in three-dimensional space is attained by the face-centered cubic packing, in which the spheres are arranged in hexagonal layers much the way that oranges are stacked at the grocery store. In August 1998, Thomas Hales announced a proof of Kepler's conjecture, obtained with his student, Samuel Ferguson. Like the proof of the four-color theorem, the proof of the Kepler conjecture reduced the problem to an extensive calculation that was then carried out by computer. Specifically, the proof shows that any counterexample to the Kepler conjecture would imply the existence of a finite arrangement of spheres satisfying certain properties, giving rise, in turn, to a certain combinatorial structure. Computer code then produced an exhaustive enumeration of the possible combinatorial structures; to be realized geometrically, any such structure would have to satisfy certain inequalities. Using branch-and-bound methods, these inequalities were relaxed to linear ones, at which point linear programming methods were used to demonstrate their infeasibility. In other words, the computations showed that there is no finite arrangement of spheres of the kind guaranteed by a putative counterexample. The proof thus consisted of a traditional mathematical argument (250 pages at the time) combined with a substantial body of computer code used to carry out the calculations.

In 1999, the *Annals of Mathematics* assigned a panel of twelve referees the task of reviewing the proof. After four years, the panel reported that they were "99% certain" that the proof was correct, but did not have the means to verify the correctness of the accompanying code. This unsatisfying state of affairs prompted Hales to embark on a project that he named "Flyspeck," to develop a computer-checked axiomatic proof.

The emerging field of *formal verification* uses logic-based computational methods to ensure the correctness of hardware and software design with respect to specifications, as well as the correctness of mathematical claims. One approach, known as *interactive theorem proving*, has users working with a computational system to construct a detailed deductive proof, starting from a small foundational system of axioms and rules. Such a formal derivation can even be checked independently of the system that constructs it. The technology needed to bridge the gap between such a low-level axiomatic presentation and an ordinary, informal mathematical proof is nontrivial, but there have already been impressive achievements along these lines. One such accomplishment is the formalization of the Feit-Thompson Odd Order Theorem by a team of researchers led by Georges Gonthier, announced in late 2012. (For surveys, see J. Avigad and J. Harrison, "Formally verified mathematics," *Communications of the ACM*, 57(4):66-75, 2014, and T. Hales, "Developments in formal proofs," *Séminaire Bourbaki*, 66éme année, no. 1086, June 2014.)

On August 10, 2014, Hales announced that he and a team of researchers had succeeded in verifying the Kepler conjecture, concluding the project he had begun in 2003. The bulk of the proof was carried out in an interactive proof system known as *HOL Light*, although the enumeration of the relevant combinatorial configurations — tame hypermaps — was verified using a system known as *Isabelle*. The book under review, *Dense Sphere Packings*, was written to support the formalization of the textual part of the proof, which Hales managed to streamline using ideas from Gonthier and Christian Marchal. (The project also made important contributions in developing methods for verifying the accompanying computations.)

As the subtitle suggests, formal verification is the subtext throughout. The book opens with colorful quotations that encourage us to reflect on the relationship between foundationalism, computation, and proof, and Hales' preface makes a strong case for the role of formal methods in mathematics: "In my view, the choice between the conventional process by a human referee and computer verification is as evident as the choice between a sundial and an atomic clock in science." An appendix summarizes the state of the verification project in May, 2012. But between the preface and appendix, the role of formal verification is left implicit, and we are given a wonderfully clear and illuminating presentation of the mathematics itself. The book is divided into three parts: the first provides an overview of the proof; the second develops the requisite background in trigonometry and measure theory, as well as key combinatorial and geometric notions (specifically, hypermaps and fans); and the third presents the central argument.

There are a number of reasons that *Dense Sphere Packings* should be of interest to members of the *Association for Symbolic Logic*. To start with, there is its historic importance: the increasing complexity of mathematical proofs and the increasing use of computers to deliver mathematical results make it inevitable that formal methods will eventually play a key role in supporting mathematical reasoning and ensuring correctness. The verification of the Kepler conjecture is an important landmark in this regard.

But the book is also notable for what it tells us about the nature of mathematical proof. The three pillars of modern logic are computability, definability, and provability, and while important strides have been made in the theory of computation and semantics, we have not progressed much beyond the basic textbook notions of theory and proof: we generally view a "theory" as a deductively closed set of sentences, and a "proof" as a sequence of assertions, each either an axiom of the theory or justified by a logical rule of inference. In reality, a mathematical theory is much more than that: it is a highly structured body of knowledge, embodying patterns of reasoning, ways of thinking, methods of problem solving, and means of calculation. A proof is similarly structured, marshaling those theoretical resources towards obtaining the desired result. Preparing the proof of the Kepler conjecture for formalization involved laying out the components with clearly demarcated interfaces and intended uses, and the resulting presentation illuminates the higher-level structure.

The best reason to spend time with the book, however, is that it presents a beautiful proof, in a manner that is clear, thoughtful, and engaging. Like any really good piece of mathematics, the result has something to tell us about the nature of mathematics itself, and there is a lot here to think about and enjoy.

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